A Phase Transition in a Markov Chain Arising from the Hopf Square Map

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A Hopf algebra can be thought of a generalization of a group. In the same way that groups generalized the Ising model, we attempt to generalize a statistical mechanics model with a Hopf algebra. In Diaconis, Pang, and Ram (2013), the Hopf square map, which combines the key operations of multiplication and comultiplication, gives rise to standard Markov chains in probability theory, like shuffling and rock breaking. However, only commutative and cocommutative Hopf algebras are used. Inspired by their work, we use the Hopf square map on a non-commutative and non-cocommutative Hopf algebra, also known as a quantum group, to give rise to a deformed walk on the natural numbers. We discover a phase transition as the deformation parameter q is varied.

I. INTRODUCTION

The Ising Model, and it's extension the Potts Model, are standard examples of statistical mechanics models where the states are elements of a group. We attempt to generalize this even further, and create a model where the states take elements of a generalization of a group: a Hopf algebra.

The second section gives a brief background on Hopf algebras, a solution to Ising Model developed by Kramers and Wannier (1941), and the Hopf square map's use in Markov chains introduced in Diaconis, Pang, and Ram (2013). The third describes the specific Hopf algebra that we work with and its important properties, and then defines the Markov chain that arises from this Hopf algebra and describes the phase transition that occurs. The fourth section then discusses future directions in this research.

II. BACKGROUND

A. Hopf algebras

1. Algebras and Coalgebras

Taken largely from Kassel (1995). The tensor product $U \otimes V$ of two vector spaces U, V is the quotient space of the Cartesian product $U \times V$ under the following relations

$$(u+u') \otimes (v+v') = (u \otimes v) + (u' \otimes v) + (u \otimes v') + (u' \otimes v')$$
$$\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$$

Bilinear maps from two vector spaces are generally linear maps of the tensor product.

The tensor product of two maps $f : U \to U'$ $g : V \to V'$ is the map $f \otimes g : U \otimes V \to U' \otimes V'$ where

$$(u \otimes v) \mapsto f(u) \otimes g(v)$$

An *algebra* is made up of a vector space A over a field k, a linear map $\mu : A \otimes A \to A$, the multiplication, and a linear map $\eta : k \to A$, the unit. The multiplication is associative, and $\eta(1)$ is an element in the center of A. These properties are shown as commutative diagrams in Figure 1. An algebra is commutative if the multiplication is commutative, which is the diagram in Figure 2. Note that τ is the flipping map, $\tau_{U,V} : U \otimes V \to V \otimes U$ by $u \otimes v \mapsto v \otimes u$.

The reason for creating the diagrams is to make the definitions for a coalgebra more intuitive: we flip the directions of all of the arrows, and put "co" in front of all of them.

The tensor product of two algebras can be an algebra as well using the natural multiplication

$$\mu_{A\otimes B}\left((a \otimes b), (a' \otimes b')\right) = \left(\mu_A(a, a') \otimes \mu_B(b, b')\right)$$

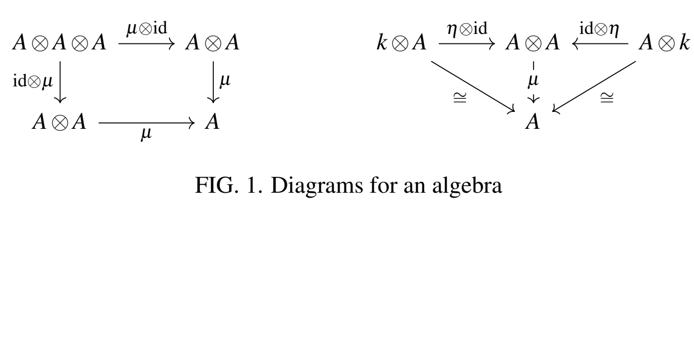

FIG. 1. Diagrams for an algebra

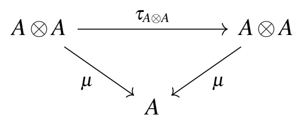

FIG. 2. Diagrams for a commutative algebra

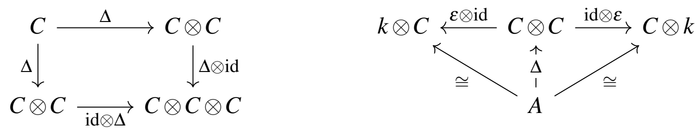

FIG. 3. Diagrams for a coalgebra

and the unit $\eta_A \otimes \eta_B$.

A morphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map $f : A \to A'$ such that

$$\mu' \circ (f \otimes f) = f \circ \mu \qquad f \circ \eta = \eta'$$

We won't use these much beyond the definition of a bialgebra. This, like most morphisms, allow us to write things like $f(a)f(b) = f(ab)$, writing multiplication without either μ.

A few examples of algebras:

1. The complex polynomials form an algebra $\mathbb{C}[x]$ with standard multiplication $\mu(p(x), q(x)) = p(x)q(x)$ and unit $\eta(\lambda) = \lambda$. This is commutative.

2. For a set $X = \{x_1, \ldots, x_n \ldots\}$, the free algebra $\mathbb{C}\{X\}$ is an algebra with basis elements (of the vector space) the words $x_{i_1} \ldots x_{i_p}$. The multiplication is the concatenation $\mu(x_{i_1} \ldots x_{i_p}, x_{j_1} \ldots x_{j_q}) = x_{i_1} \ldots x_{i_p} x_{j_1} \ldots x_{j_q}$, extended linearly for the elements. The unit is the empty word $\emptyset$, $\eta(\lambda) = \lambda\emptyset$.

3. Given a group G we can form the vector space $\mathbb{C}G$ of complex linear combinations $\sum_{g\in G} c_g g$, which forms an algebra. The multiplication is defined by the group operation, $\mu\left(\sum_g c_g g, \sum_h d_h h\right) = \sum_{g\in G}\left(\sum_{k\in G} c_{gk^{-1}} d_k\right) g$. The unit is the identity element, $\eta(\lambda) = \lambda e$. This is generally not commutative.

4. Similarly, the functions on a finite group $\mathbb{C}^G$ form an algebra with pointwise multiplication: $\mu(f, h)$ is the function that sends $g \mapsto f(g)h(g)$ and unit is the constant function $\eta(\lambda) = \lambda\mathbb{1}$, where the constant function $\mathbb{1}(g) = 1$. This is commutative.

A *coalgebra* is a vector space C over a field k with a linear map $\Delta : C \to C \otimes C$, the comultiplication, and $\varepsilon : C \to k$, the counit, that are coassociative and counital. These definitions are the commutative diagrams in Figure 3. Cocommutativity is the commutative diagram in Figure 4.

The tensor product of two coalgebras uses the comultiplication $(\mathrm{id} \otimes \tau_{C_1,C_2} \otimes \mathrm{id}) \circ (\Delta_1 \otimes \Delta_2)$ and counit $\varepsilon_1 \otimes \varepsilon_2$

A morphism of coalgebra is similarly defined as

$$\Delta' \circ f = (f \otimes f) \circ \Delta \qquad \varepsilon = f \circ \varepsilon'$$

1. Given a group G we can form a coalgebra $\mathbb{C}G$. The vector space is the same as the algebra above. The comultiplication is defined by $\Delta(g) = g \otimes g$ and extended linearly, while the counit is the identity, $\varepsilon(g) = 1$. This is cocommutative. Elements of a coalgebra who have coproduct $\Delta(x) = x \otimes x$ are called *group-like*.

2. The functions on a finite group $\mathbb{C}^G$ form a coalgebra with coproduct $\Delta(f)(g \otimes h) = f(gh)$ (remember that $\Delta(f) \in \mathbb{C}^G \otimes \mathbb{C}^G$) and counit the evaluation at the identity, $\varepsilon(f) = f(e)$. This is not cocommutative.

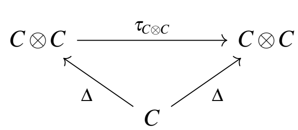

FIG. 4. Diagrams for a cocommutative coalgebra

3. On a set X, the free algebra $\mathbb{C}\{X\}$ is a coalgebra with basis elements (of the vector space) the words $x_{i_1}\cdots x_{i_p}$. There is a comultiplication defined by

$$\Delta\left(x_{i_1}\cdots x_{i_p}\right) = 1\otimes x_{i_1}\cdots x_{i_p} + \sum_{j=1}^{p} x_{i_1}\cdots x_{i_j}\otimes x_{i_{j+1}}\cdots x_{i_p} + x_{i_1}\cdots x_{i_p}\otimes 1$$

   This is **not** the standard coproduct and will not be used further. The counit sends all non-trivial words to 0, and the trivial word to 1.

The dual of a coalgebra (C,Δ,ε) is an algebra. The vector space is C^*. Defining $\sigma_{U,V} : U^*\otimes V^* \to (V\otimes U)^*$, $(\sigma(f\otimes g))$ as the map sending $v\otimes u \mapsto f(u)\otimes g(v)$ (the switching is useful for technical results), the multiplication is $\Delta^*\circ\sigma_{C,C}\circ\tau_{C^*,C^*}$. And the unit is ε^* (the $*$ on a vector space is its dual, on a linear map is its transpose).

When the vector space A is finite dimensional, the map $\sigma_{A^*,A^*} : A^*\otimes A^* \to (A\otimes A)^*$ is an isomorphism. This means that the dual of a finite dimensional algebra (A,μ,η) is a coalgebra with vector space A^*, comultiplication $\tau_{A^*,A^*}\circ\sigma_{A,A}^{-1}\circ\mu^*$ and counit η^*.

We must mention Sweedler's notation for the coproduct: in the tensor of two vector spaces, $U\otimes V$, any element is of the form

$$\sum_{i=1}^{p} u_i\otimes v_i \tag{1}$$

where the $u_i\in U$ and $v_i\in V$ for all $1\le i\le p$. This is always a finite sum as $C\times C$ is a vector space. For a coalgebra C and its coproduct Δ, since $\Delta(x)\in C\otimes C$, then

$$\Delta(x) = \sum_{i=1}^{p} x_i'\otimes x_i''$$

This finite sum is common enough that we will often abbreviate it either as

$$\Delta(x) = \sum_{(x)} x'\otimes x'' = \sum_{(x)} x_{(1)}\otimes x_{(2)} \tag{2}$$

But this allows many properties of the coalgebra to be written out explicitly. For example, coassociativity reformulated in terms of equation 2 means that for any $x\in C$,

$$\left((\Delta\otimes \mathrm{id}_C)\circ\Delta\right)(x) = \sum_{(x)}\left(\sum_{(x_{(1)})}(x_{(1)})_{(1)}\otimes(x_{(1)})_{(2)}\right)\otimes x_{(2)} = \sum_{(x)} x_{(1)}\otimes\left(\sum_{(x_(2))}(x_{(2)})_{(1)}\otimes(x_{(2)})_{(2)}\right) = \left((\mathrm{id}_C\otimes\Delta)\circ\Delta\right)(x)$$

Therefore we identify the two equal sums as $\sum_{(x)} x_{(1)}\otimes x_{(2)}\otimes x_{(3)}$ and could continue with associativity in any entry of the tensor product.

Counitality, cocommutativity, and morphisms of coalgebras can all be restated using Sweedler's notation in equation 2, and the explicit expressions simplify many proofs involving coalgebras.

2. Bialgebras and Hopf algebras

A vector space H can simultaneously be given an algebra and a coalgebra structure. Especially interesting to us is when these structures are compatible: when μ and η are morphisms of coalgebras or, equivalently, Δ and ε are morphisms of algebras. We usually check the second condition, that is

$$\begin{aligned} \mu\circ(\Delta\otimes\Delta) &= \Delta\circ\mu \qquad & \Delta\circ\eta &= \eta\otimes\eta \\ \mu\circ(\varepsilon\otimes\varepsilon) &= \varepsilon\circ\mu \qquad & \varepsilon\circ\eta &= \mathrm{id} \end{aligned}$$

Which for each element x,y means that (as all of these are linear maps)

$$\begin{aligned} \Delta(x)\Delta(y) \equiv \mu(\Delta(x),\Delta(y)) &= \Delta(\mu(x,y)) \equiv \Delta(xy) \qquad & \Delta(\eta(1)) &= \eta(1)\otimes\eta(1) \\ \varepsilon(x)\varepsilon(y) \equiv \mu(\varepsilon(x),\varepsilon(y)) &= \varepsilon(xy) \equiv \varepsilon(\mu(x,y)) \qquad & \varepsilon(\eta(1)) &= 1 \end{aligned}$$

(The $\equiv$ signs are writing out the multiplication in the algebra explicitly, the $=$ signs are the properties to check). In Sweedler's notation, the morphism property for the coproduct is

$$\sum_{(x)(y)} x'y' \otimes x''y'' = \sum_{(xy)} (xy)' \otimes (xy)''$$

A bialgebra is a vector space H that is an algebra and coalgebra with these compatibility conditions. Bialgebras are commutative if the algebra is, and similarly cocommutative if the coalgebra is.

1. The algebra and coalgebra structures on $\mathbb{C}G$ form a bialgebra. The comultiplication is a morphism of algebras:

$$\Delta(x)\Delta(y) = (x\otimes x)(y\otimes y) = (xy\otimes xy) = \Delta(xy) \qquad \Delta(\eta(1)) = \Delta(e) = e\otimes e = \eta(1)\otimes\eta(1)$$

as is the counit

$$\varepsilon(x)\varepsilon(y) = 1\cdot 1 = \varepsilon(xy) \qquad \varepsilon(\eta(1)) = \varepsilon(e) = 1$$

2. The functions on a finite group $\mathbb{C}^G$ also form a bialgebra, since for every $g,h\in G$,

$$\Delta(f)\Delta(h)(g_1\otimes g_2) = f(g_1g_2)h(g_1g_2) = (fh)(g_1g_2) = \Delta(fh)(g_1\otimes g_2)$$

$$\Delta(\eta(1))(g_1\otimes g_2) = \Delta(\mathbb{1})(g_1\otimes g_2) = \mathbb{1}(g_1g_2) = 1 = \eta(1)\otimes\eta(1)(g_1\otimes g_2)$$

$$\varepsilon(f)\varepsilon(g) = f(e)g(e) = (fg)(e) = \varepsilon(xy) \qquad \varepsilon(\eta(1)) = \varepsilon(\mathbb{1}) = \mathbb{1}(e) = 1$$

which makes them both morphisms of algebras.

3. The free associative algebra (or isomorphically, the tensor algebra on a suitable vector space) on a set $X = \{x_1,\ldots,x_n,\ldots\}$, $\mathbb{C}\{X\}$ is a bialgebra. The algebra structure is the same as above. The coproduct is defined on the elements of X and extended by making it a linear map and an algebra morphism: $\Delta(x) = 1\otimes x + x\otimes 1$ for all $x\in X$. Elements with this coproduct are called *primitive* elements.

   The coproduct on words of length n involves the shuffles in the symmetric group, or, equivalently, since the coproduct must be an algebra morphism,

$$\begin{aligned}\Delta(x_{i_1}\cdots x_{i_p}) &= \Delta(x_{i_1})\cdots\Delta(x_{i_p})\\ &= (x_{i_1}\otimes 1 + 1\otimes x_{i_1})\cdots(x_{i_p}\otimes 1 + 1\otimes x_{i_p})\\ &= \sum_{S\subseteq[n]}\left(\prod_{k\in S} x_{i_k}\right)\otimes\left(\prod_{k\in[n]\setminus S} x_{i_k}\right)\end{aligned}$$

   As when distributing we choose either the first term or second term in each factor, so for any subset S of $[n] = \{1,2,\ldots,k\}$, we can get $\prod_{k\in S} x_{i_k}$ in the first slot of the tensor product, and its compliment in the second. Of course, order matters in these products, so the indices go through the sets in increasing order.

   The counit is defined by $\varepsilon(\lambda) = \lambda$ and $\varepsilon(x) = 0$ for all $x\in X$, implying $\varepsilon(w) = 0$ for all $w\in\mathbb{C}\{X\}\setminus\mathbb{C}$. This is a non-commutative, but cocommutative bialgebra.

By the above results, the dual of a finite dimensional bialgebra is also a bialgebra.

For an algebra A and coalgebra C, if f,g are linear maps from $C\to A$, the convolution is the bilinear map

$$f\star g = \mu\circ(f\otimes g)\circ\Delta \tag{3}$$

For a bialgebra H, the convolution $\star$ is defined on the vector space of endomorphisms of H.

The convolution can be the multiplication of the algebra of linear maps from $C\to A$; it is associative; $(f\star g)\star h = f\star(g\star h)$. And it has a unit $\eta\varepsilon$; for one side, note

$$\begin{aligned}((\eta\varepsilon)\star f)(x) &= \textstyle\sum_{(x)}\varepsilon(x')\eta(1)f(x'')\\ &= \textstyle\sum_{(x)} f(\varepsilon(x')(x''))\\ &= f(x)\end{aligned}$$

The first equality following from Sweedler's notation and the definition of the convolution, the second because $\varepsilon(x') \in k$, $\eta(1)$ is in the center, and f is linear, and the third from counitality. The other side is proved similarly, and thus

$$(\eta\varepsilon)\star f = f\star(\eta\varepsilon) = f$$

An endomorphism S of a bialgebra H is an antipode if

$$S\star \mathrm{id}_H = \mathrm{id}_H \star S = \eta\circ\varepsilon \tag{4}$$

Under the convolution operation, the antipode is the inverse for the identity map.

A *Hopf algebra* is a bialgebra with an antipode. We are most interested in non-commutative and non-cocommutative Hopf algebras, sometimes called *quantum groups*.

Antipodes have many different properties:

1. If a bialgebra has an antipode (which is not guaranteed), that is the only one it can have.

   If S and S' are antipodes, then since $\eta\varepsilon$ is a unit for $\star$ and it is associative,

   $$S = S\star(\eta\varepsilon) = S\star(\mathrm{id}_H\star S') = (S\star\mathrm{id}_H)\star S' = (\eta\varepsilon)\star S' = S' \tag{5}$$

2. S^* satisfies the properties of an antipode on the dual H^* (though this might not be a bialgebra)

3. For all $x,y\in H$ a Hopf algebra, $S(xy) = S(y)S(x)$ and $S(1) = 1$

   In fact, if f is a linear map on H such that the above holds and f satisfies the defining property of an antipode on a generating set of H, then f is an antipode.

4. $(S\otimes S)\circ\Delta = \tau_{H,H}\circ\Delta\circ S$ and $\varepsilon\circ S = \varepsilon$

Some examples of Hopf algebras:

1. We have already seen that $H = \mathbb{C}G$ is a bialgebra, and we are yet to use the inverse property of the group. Define a map $S(g) = g^{-1}$. Then for any $g\in G$

   $$\begin{aligned}(S\star\mathrm{id}_H)(g) &= (\mu\circ S\otimes\mathrm{id}_H\circ\Delta)(g)\\ &= (\mu\circ S\otimes\mathrm{id}_H)(g\otimes g)\\ &= S(g)\mathrm{id}_H(g)\\ &= g^-1g = e\\ &= \eta(1)\\ &= \eta(\varepsilon(g))\end{aligned}$$

   and similarly for the other side. Thus, S is an antipode, and $\mathbb{C}G$ is a Hopf algebra.

2. When H is a finite dimensional Hopf algebra, then the dual is also both an algebra and a coalgebra, and H^* has an antipode S^*, so it is also a Hopf algebra. The functions on the group then also have a Hopf algebra structure.

3. The free algebra $\mathbb{C}\{X\}$ is a Hopf algebra where the antipode is defined on the unit 1 and words $x_{i_1}x_{i_2}\dots x_{i_p}$ by

   $$S(1) = 1 \qquad S(x_{i_1}x_{i_2}\dots x_{i_p}) = (-1)^p x_{i_p}\dots x_{i_2}x_{i_1}$$

   and extended linearly.

A graded Hopf algebras is one of the form $H = \bigoplus_{i=0}^{\infty} H_i$, where H_0 is the base field and each H_i is a vector space, the multiplication and comultiplication respects the grading:

$$x_i\in H_i,\quad x_j\in H_j \implies \mu(x_i,x_j)\in H_{i+j} \qquad \Delta(x_i)\in\bigoplus_{j=0}^{i} H_j\otimes H_{i-j}$$

and if $i>0$, $\varepsilon(x_i) = 0$. The elements of H_i are homogeneous of degree i.

The free algebra on n variables is a graded Hopf algebra, where each H_i is made up of the words of length i. Each of the n variables is itself degree 1, and the basis for each H_i are the monomials of degree i.

3. Hopf Algebras from Lie Algebras

One more important kind of Hopf algebra comes from any Lie algebra. A Lie algebra is vector space L with a bilinear map called the Lie bracket $[\cdot,\cdot] : L \times L \to L$. This map is anti-symmetric– flipping the inputs flips the sign of the sign of the output– and satisfies a Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

Any Lie algebra can be made into a Hopf algebra by taking the free associative algebra on the basis elements of L. Take the ideal of the free associative algebra generated by the relations $[x,y] = xy - yx$ for all $x,y \in L$, and the resulting quotient $U(L)$ is called the *enveloping algebra*. Since the free associative algebra is always cocommutative, so is the enveloping algbera.

As in the free associative algebra, the elements of L have coproduct $\Delta(x) = 1 \otimes x + x \otimes 1$, so these are all primitive elements. The full coproduct is extended linearly, as above. The counit sends all of the elements to 0, and the antipode is again $S(x_{i_1} x_{i_2} \dots x_{i_p}) = (-1)^p x_{i_p} \dots x_{i_2} x_{i_1}$.

Note that for any two primitive elements $x,y \in L$, the product may not be primitive:

$$\Delta(xy) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) = 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1$$

But the commutator is:

$$\begin{aligned} \Delta(xy - yx) = \Delta(xy) - \Delta(yx) &= 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1 - 1 \otimes yx - y \otimes x - x \otimes y - yx \otimes 1 \\ &= 1 \otimes xy - 1 \otimes yx + xy \otimes 1 - yx \otimes 1 = 1 \otimes (xy - yx) + (xy - yx) \otimes 1 \end{aligned}$$

Since the commutator operation is always antisymmetric and satisfies the Jacobi identity, then the primitive elements of a Hopf algebra always form a Lie algebra with the commutator as the Lie bracket.

A key theorem for enveloping algebras is the Poincaré-Birkoff-Witt theorem: if $\{x_1, x_2, \dots\}$ is a basis for the Lie algebra L, then the symmetric products

$$\left\{ \sum_{\sigma \in S_k} x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \cdots x_{i_{\sigma(k)}} | 1 \le i_1 \le \cdots \le i_k, k \in \mathbb{N} \right\}$$

form a basis for the enveloping algebra $U(L)$.

B. Ising and Potts Models

The Ising Model is the standard example in statistical mechanics of ferromagnetism. We adapt Kramers and Wannier (1941).

We start with N states that can be thought of either in a line with N significantly large such that end effects are insignificant, or in a circle so that the first and last state are side by side. We label the states σ_i, $1 \le i \le N$, and each state σ_i has a connection to the state preceding, σ_{i-1} and after it σ_{i+1}.

The states can be labeled in $\mathbb{Z}_2$, so each $\sigma_i = \pm 1$, representing spin up (1) or down (-1) for each state. We can describe a labeling of the entire system by $\vec{\sigma} = (\sigma_1, \dots, \sigma_N)$. The energy of the system depends only on the nearest neighbor interactions– the connections mentioned before. Each interaction contributes the same: a positive constant if they are aligned, and that negative constant if they are in opposite directions. The Hamiltonian of the system is

$$H(\vec{\sigma}) = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} \tag{6}$$

where $\sigma_{N+1} = \sigma_1$ based on our assumption that our states form a circle, β is proportional to the inverse of temperature, and J is a constant.

Finding the free energy of the system involves finding the partition function

$$Z = \sum_{\vec{\sigma}} e^{-\beta H(\vec{\sigma})} \tag{7}$$

a sum over all of the possible labelings of the N states; in this case, 2^N of them. Explicitly, $\sum_{\vec{\sigma}} = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1}$

Then the free energy takes the number of states to infinity,

$$A = \lim_{N \to \infty} \frac{\ln Z}{N} \tag{8}$$

But that log gets relatively ugly. However, we can use a transition matrix

$$T = \begin{bmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{bmatrix} \tag{9}$$

Representing our two states $|1\rangle$ and $|-1\rangle$ as an orthonormal basis, then we have that for any $\sigma_i, \sigma_j = \pm 1$

$$\langle \sigma_i | T \left| \sigma_j \right\rangle = e^{\beta J \sigma_i \sigma_j}$$

two facts that can break down the partition function,

Then, since it is an orthonormal basis, $\sum_{a=\pm 1} |a\rangle \langle a| = \mathbb{I}_2$, and thus

$$Z = \sum_{\vec{\sigma}} \prod_{i=1}^{N} \langle \sigma_i | T | \sigma_{i+1} \rangle = \sum_{\sigma_1} \langle \sigma_1 | T^N | \sigma_1 \rangle$$

Again using that $\sigma_{N+1} = \sigma_1$. That last sum is the trace of the matrix T^N, which is the sum of its eigenvalues. But the eigenvalues of T^N are the eigenvalues of T, λ_1 and λ_2, raised to that same power:

$$Z = \lambda_1^N + \lambda_2^N$$

Thus, if $\lambda_1 > \lambda_2$,

$$A = \lim_{N\to\infty} \frac{\ln\left(\lambda_1^N + \lambda_2^N\right)}{N} = \lim_{N\to\infty} \frac{\ln\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) + N\ln(\lambda_1)}{N} = \ln \lambda_1$$

For our specific matrix T, the eigenvalues are

$$e^{\beta J} \pm e^{-\beta J}$$

So the free energy is $\ln(e^{\beta J} + e^{-\beta J})$. Here, there is no phase transition as this is continuous.

It can be shown, numerically, that the two dimensional Ising model does have a phase transition. We will not go into that here.

The Pott's model is a generalized version of the Ising model, where $\mathbb{Z}_2$ is replaced by another cyclic group $\mathbb{Z}_n$. The Hamiltonian must be changed; instead of multiplying the two spins the most standard replacement is the Kronecker delta

$$H(\vec{\sigma}) = -J \sum_{i=1}^{N} \delta(\sigma_i, \sigma_{i+1})$$

This has a similar effect as the multiplication of the up and down spins ± 1 in the Ising Model. In two dimensions, the Pott's model also has a phase transition.

The group does not have to be cyclic, necessarily. Any group could be used as the labeling of the states in the Ising Model, and the Hamiltonian becomes a function on those group elements. But as we have already seen, the functions on a group are a Hopf algebra, so maybe we should try to examine Hamiltonians and partition functions in a Hopf algebra.

C. The Hopf Square Forming a Markov Chain

A more direct use of Hopf algebra's in this area was Diaconis, Pang, and Ram (2013) application to well-known Markov chains.

The linear map on a Hopf algebra H formed by the composition $\Psi^2 := \mu \circ \Delta$ can be thought of as a process that sends elements of the Hopf algebra to others. That is, from equation 2, $\Delta(x)$ is always of the form $\sum_{(x)} x_{(1)} \otimes x_{(2)}$, so $\Psi^2(x) = \sum_{(x)} x_{(1)} x_{(2)}$ is a sum of elements in the Hopf algebra. In a graded Hopf algebra, both the multiplication and comultiplication respect the grading, and thus the Hopf square map respect the grading. Thus, each grading can be looked at individually. They found that writing all of the elements in terms of a natural basis leads to an interpretation as a Markov chain. The properties of types of Hopf algebra's they used help to find the eigenvalues and vectors of these Markov chains, which allows for the answers of probabilistic questions.

A simple definition that is enough for our purposes: a Markov chain on a set S comes from the probability of moving from state x to state y, $T(x,y) \geq 0$, for all $x, y \in S$. A Markov chain is a sequence of random variables $X_0, X_1, \ldots$ taking values in S, starting in a fixed state $X_0 = x_0$, such that the probability that $X_n = x_n$ depends only on the value of X_{n-1} according to T,

$$\mathbb{P}(X_n = x_n | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}) = T(x_{n-1}, x_n)$$

And thus

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=0}^{n-1} T(x_i, x_{i+1})$$

This means that fixing a state x, $\sum_{y \in S} T(x,y) = 1$ and is a probability distribution. T is also called transition matrix.

A key result of their work is the following.

Theorem 1. *Let $H = \bigoplus H_n$ be a graded Hopf algebra over $\mathbb{R}$ which, as an algebra, is either a polynomial algebra or a cocommutative free associative algebra.*

Then there is a basis $\mathfrak{B}_n$ of H_n such that the matrix representation of the normalized Hopf square $\frac{1}{2^n}\Psi^2$ in that basis, transposed, is a transition matrix. That is,

$$\frac{1}{2^n}\Psi^2(b) = \sum_{b' \in \mathfrak{B}_n} T_n(b,b')b'$$

Satisfies

$$T_n(b,b') \geq 0, \sum_{b' \in \mathfrak{B}_n} T_n(b,b') = 1$$

$\mathfrak{B}_n$ is a rescaling of the monomials/words of the generating variables.

They describe that for Hopf algebras that as an algebra are either polynomials in commuting variables or words in non-commuting variables and cocommutative (a free associative algebra), there is a re-scaling of the standard basis that allows the Hopf Square Ψ^2 to give rise to a transition matrix through the coefficients.

This basis is constructed explicitly from the basis of monomials in the generators, with other assumptions on the coefficients.

Finding the spectrum is useful for analyzing the Markov chain. The Hopf algebra structure helps there, too. See that if an element x of a Hopf Algebra H is primitive (that $\Delta(x) = 1 \otimes x + x \otimes 1$), then x is an eigenvector of the Hopf square with eigenvalue 2, as

$$\Psi^2(x) = m \circ \Delta(x) = m(1 \otimes x + x \otimes 1) = m(1 \otimes x) + m(x \otimes 1) = 2x$$

Then taking k primitive elements $x_1, \ldots, x_k$ of H, it follows the symmetric product $\sum_{\sigma \in S_k} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}$ is an eigenvector with eigenvalue 2^k: see that from linearity and the coproduct of the free algebra, we have

$$\begin{aligned} m \circ \Delta\left(\sum_{\sigma \in S_k} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}\right) &= m\left(\sum_{\sigma \in S_k} \Delta\left(x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}\right)\right) \\ &= m\left(\sum_{\sigma \in S_k} \sum_{A \subseteq [k]} \prod_{i \in A} x_{\sigma(i)} \otimes \prod_{j \in A^c} x_{\sigma(j)}\right) \end{aligned}$$

By applying the multiplication map to the tensor product, we get

$$= \sum_{\sigma \in S_k} \sum_{A \subseteq [k]} \prod_{i \in A} x_{\sigma(i)} \prod_{j \in A^c} x_{\sigma(j)}$$

We realize that each term in the sum $\sum_{A \subseteq [k]}$ is a distinct permutation of the original elements, and because each A is distinct and we sum over all subsets of $[k]$, we get each and every element of S_k. Then each term in the outer sum is the same, and we get

$$= |\{A | A \subseteq [k]\}| \sum_{\sigma \in S_k} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}$$

Then there are 2^k ways to pick the subset A, so the eigenvalue is 2^k,

$$\Psi^2\left(\sum_{\sigma \in S_k} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}\right) = 2^k \sum_{\sigma \in S_k} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)} \tag{10}$$

Remembering the Poincaré-Birkoff-Witt theorem from Lie algebras, the elements of a Lie algebra are primitive in the enveloping algebra and the symmetric products of these elements form a basis for the enveloping algebra. Thus, if we have an ordered basis for the Lie algebra L, then we have an eigenbasis for the Hopf square on the enveloping algebra.

But even further, remembering the Lie algebra of primitive elements from a Hopf algebra H, the Cartier-Milnor-Moore theorem states that if H is a graded, cocommutative, and connected Hopf algebra, then the enveloping algebra of this Lie algebra of primitives is isomorphic to H. So if our Hopf algebra satisfies the above conditions, finding an eigenbasis amounts to finding a basis for the primitives of that Lie algebra– not a small task, but a worthwhile simplification.

In both the commutative and cocommutative cases, they use the Eulerian idempotent map

$$e(x) = \sum_{a\geq 1} \frac{(-1)^{a-1}}{a} \mu^a \circ \bar{\Delta}^a(x)$$

Where $\bar{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$, and for both μ^a and $\bar{\Delta}^a$, $f^a = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f) \circ f^{a-1}$, using the associativity or coassociativity of the Hopf algebra. This can be shown to be a projection onto the 2-eigenspace of Ψ^2 for commutative Hopf algebras. For graded, cocommutative Hopf algebras, it projects onto the primitive elements, which amounts to the same 2-eigenspace. So this map is a key component of finding the spectrum of the derived Markov chains. However, we want to work with Hopf algebras that are not cocommutative, so this theory will not apply to our work.

A brief recap of some of the examples in the paper:

1. The free algebra $H = \mathbb{C}\{x_1, \ldots, x_n\}$ with non-commuting generators, with concatenation as the multiplication and the comultiplication on the generators is $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$. This is cocommutative.

   Each basis element of length k (for example, $x_1x_2x_3$) is sent by the Hopf square to a sum of 2^k other basis words (for example, 2^3), each with coefficient 1. When normalized by the number of words, and each word representing an ordering of a deck of k cards, this becomes the Markov chain for the Gilbert-Shannon-Reeds inverse shuffle.

   The dual Hopf algebra is the usual model for riffle shuffling.

2. The Hopf algebra of symmetric polynomials in n (commuting) variables $\Lambda(x_1, \ldots, x_n)$ has a basis of the elementary symmetric polynomials

   $$e_k := \sum_{1\leq j_1 < \ldots < j_k \leq n} x_{j_1} \cdots x_{j_k}$$

   The product is the usual multiplication of polynomials and the coproduct defined on this basis is

   $$\Delta(e_k) = k! \sum_{i=0}^{k} e_i \otimes e_{k-i}$$

   Applying the Hopf square map gives the rock-breaking Markov chain.

The assumptions that the paper makes are not extensive, but largely eliminate many of the Hopf algebras that we would like to study: true quantum groups that are non-commutative and non-cocommutative Hopf algebras. They use only graded Hopf algebras, and assume that each H_i are finite dimensional.

III. EXTENSION

A. The Hopf algebra

We attempt to apply the Hopf Square map to a quantum group. One of the most simple non-commutative Lie algebras is $\mathfrak{sl}(2)$, the traceless 2×2 matrices generated by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With the commutators $[X,Y] = H$, $[H,X] = 2X$, and $[H,Y] = -2Y$.

We will end up working with the analogue of the Lie algebra generated by just H, X, as this is still closed. This is is the Lie Algebra for the Borel subgroup of 2×2 matrices $\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}$. We know the enveloping algebra of a Lie algebra always forms a cocommutative Hopf algebra. Thus, we use a q-deformation to make a non-commutative and non-cocommutative Hopf algebra. We follow Kassel in constructing the one parameter deformation U_q.

For $q \neq \pm 1$ in $\mathbb{R}$ (but really any field), U_q as an algebra is generated by E, F, K, K^{-1} with the multiplication relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KE = q^2 EK, &\qquad KF = q^{-2}FK \\ [E,F] = EF &- FE = \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

The last fraction is defined since $q \neq q^{-1}$. The set $\left\{E^i F^j K^l\right\}_{i,j \in \mathbb{N}, l \in \mathbb{Z}}$ is a basis. The parameter q in these relations makes the algebra non-cocommutative. If $q \to 1$, then U_1 is isomorphic to then universal enveloping algebra $U(\mathfrak{sl}(2))[K]/(K^2 - 1)$ via

$$E \mapsto XK, \quad F \mapsto Y, \quad K \mapsto K, \quad \frac{K - K^{-1}}{q - q^{-1}} \mapsto HK$$

To make this a Hopf algebra, the comultiplication, counit, and antipode are

$$\begin{aligned} \Delta(E) = 1 \otimes E + E \otimes K &\qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1 \\ \Delta(K) = K \otimes K &\qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1} \\ \varepsilon(E) = \varepsilon(F) = 0 &\qquad \varepsilon(K) = \varepsilon(K^{-1}) = 1 \\ S(E) = -EK^{-1} \quad S(F) = -KF &\qquad S(K) = K^{-1} \quad S(K^{-1}) = K \end{aligned}$$

These satisfy all of the axioms of a Hopf algebra, and importantly, is neither commutative or cocommutative: this is a true quantum group.

For the simplest Markov chain to examine, we ignore F; the sub-Hopf algebra B_q is generated by E, K is closed. Our new basis is $\left\{E^i K^l\right\}_{i \in \mathbb{N}, l \in \mathbb{N}}$, and this is a vector space graded by the power of E. Each vector space $\left\langle E^i K^l | l \in \mathbb{N}\right\rangle$ is infinite dimensional, but their direct sum is the full Hopf algebra.

For any basis element, we can use the commutation relations to give a formula for the commultiplication.

$$\Delta(E^i K^l) = \sum_{r=0}^{i} q^{r(i-r)} \left[\begin{smallmatrix} i \\ r \end{smallmatrix}\right] E^{i-r} K^l \otimes E^r K^{l+(i-r)} \tag{11}$$

The symmetric q-binomial coefficients $\left[\begin{smallmatrix} i \\ r \end{smallmatrix}\right]$, interesting in their own right, are defined progressively as follows: since q is not a root of unity, we have the well-defined q-numbers

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

These q-numbers have symmetric properties as $[-n] = -[n]$ and $[m+n] = q^n[m] + q^{-m}[n]$. and are used to define q factorials: $[0]! = 1, [k]! = [k][k-1]\cdots[2][1]$, and for $0 \leq k \leq n$ the q binomial coefficients

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \frac{[n]!}{[k]![n-k]!}$$

One way this is useful is for non-commuting variables such that $yx = q^2 xy$, we have

$$(x+y)^n = \sum_{k=0}^{n} q^{k(n-k)} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] x^k y^{n-k}$$

B. The Markov chain

With a formula in equation 11 for the comultiplication, we can use the Hopf square map to work towards a Markov chain.

$$\Psi^2(E^i K^l) = \sum_{r=0}^{i} q^{r(i-r)} \left[\begin{smallmatrix} i \\ r \end{smallmatrix}\right] E^{i-r} K^l E^r K^{l+(i-r)} = \sum_{r=0}^{i} q^{r(i-r+2l)} \left[\begin{smallmatrix} i \\ r \end{smallmatrix}\right] E^i K^{2l+i-r}$$

Where we have used the commutation relation $K^n E^m = q^{2mn} E^m K^n$. For fixed i, the basis elements of the grading can be labeled by the power of K. So the integers will be our states, and the Markov chain will start in state $l = 0$.

The $i=0$ grading has a trivial Markov chain: since

$$\Psi^2(K^l)=K^{2l}$$

then the transition probability is

$$P(l,j)=\delta(2l,j)$$

And otherwise 0. The state l moves to the state $2l$ with probability 1, which isn't particularly interesting, so we move to the $i=1$ grading.

$$\Psi^2\left(EK^l\right)=EK^{2l+1}+q^{2l}EK^{2l} \tag{12}$$

In this process, the $l\to 2l$ process always occurs but there is some randomness between moving $2l\to 2l$ or $2l\to 2l+1$. We can break up the Hopf Square map into a composition, the determined part of the process and the random part. The function D that sends $EK^l\mapsto EK^{2l}$ is determined, and the random part is the function Φ_1 that maps $EK^l\mapsto EK^l+q^lEK^l$. See that on the $i=1$ grading, $\Psi^2=\Phi_1\circ D$. To focus on just the randomness in the process, we focus on

$$\Phi_1(EK^l)=EK^{l+1}+q^lEK^l \tag{13}$$

We will also normalize so that the coefficients add to 1, though we have to do so row by row instead of the whole matrix at once. We are left with the process on the integers l (the powers of K), where the probability of transition is

$$P(l,j)=\begin{cases}\frac{1}{q^l+1} & j=l+1\\ \frac{q^l}{q^l+1} & j=l\\ 0 & \text{else}\end{cases} \tag{14}$$

At this point, to make these probabilities, we make sure that $q\geq 0$. Since our starting state is $l=0$ and we can only move towards more positive states, we never reach negative values. Thus, our transition matrix, indexed from 0, is

$$T=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \cdots\\ 0 & \frac{q^2}{1+q^1} & \frac{1}{1+q^1} & 0 & 0 & \cdots & 0 & \cdots\\ 0 & 0 & \frac{q^2}{1+q^2} & \frac{1}{1+q^2} & 0 & \cdots & 0 & \cdots\\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \cdots\\ 0 & 0 & \cdots & 0 & \frac{q^n}{1+q^n} & \frac{1}{1+q^n} & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\end{bmatrix} \tag{15}$$

Now, we would like to have a probability distribution on the Markov chain: starting with $X_0=0$, what is the probability that $X_t=k$? That is, after t time steps the chain moved to state k. Since each state has only two options, staying still (a "failure") and a move to the right (a "success"), a location is never repeated. Therefore we are looking for the probability of k successes in t trials, but the probability of failure changes at each location. We will have exactly k successes, one at each location, which must always contribute

$$\left(\frac{1}{q^0+1}\right)\left(\frac{1}{q^1+1}\right)\cdots\left(\frac{1}{q^{(k-1)}+1}\right):=((-1;q)_k)^{-1}$$

Where $(a;q)_k$ is the q-analogue of the Pochamer symbol, defined to be $(1-a)(1-aq)\cdots(1-aq^{n-1})$.

We then have to account for the $t-k$ failures. They can occur at any of the $k+1$ locations $\{0,\dots,k\}$, with any number of failures at any location. This is a partition of $t-k$ into $k+1$ parts.

$$y_0+y_1+\cdots+y_k=t-k$$

Where y_i indicates the number of failures at state i, $0\leq i\leq k$. The probability of a specific partition of the failures is

$$\left(\frac{q^0}{q^0+1}\right)^{y_0}\left(\frac{q^1}{q^1+1}\right)^{y_1}\cdots\left(\frac{q^k}{q^k+1}\right)^{y_k}$$

Thus, the likelihood of having exactly k successes in t trials is the sum of these probabilities

$$\mathbb{P}(X_t=k)=\frac{1}{(-1;q)_k}\left[\sum_{y_0+\cdots+y_k=t-k}\prod_{i=0}^{k}\left(\frac{q^i}{q^i+1}\right)^{y_i}\right] \tag{16}$$

C. Phase Transition

The Markov chain constructed in the previous section depends heavily on the parameter q.

1. When $q = 1$, then all of the probabilities are exactly $\frac{1}{2}$, and the distribution is binomial:

$$\mathbb{P}(X_t = k) = \binom{t}{k}\frac{1}{2^t}$$

which means that the expectation $\mathbb{E}(X_t) = \frac{t}{2}$.

2. When $q = 0$, there is a fair coin flip at position 0, but once the chain has moved beyond that, the probability of success is always 1. Thus, the expectation will be on the order of t.

3. As $q \to \infty$, the fair coin flip remains at position 0, but afterwards, the probability of success approaches 0. If we take those probabilities to be exactly 0, then the expectation is the probability that the chain has reached position 1, $\mathbb{E}(X_t) = 1-(\frac{1}{2})^t$.

This suggests that a phase transition occurs at $q = 1$, and plots of the expectation agree: when $q > 1$, the expectation is sub-linear, while when $0 \le q < 1$, the expectation is linear. We prove this formally for any chain that has our general characteristics.

Theorem 2. *Let $X_{t,\alpha}$ it a Markov chain $X_0, X_1, \ldots$ on the natural numbers $\{0,1,2,\ldots,\}$ that depends only on the failure probabilities $\alpha_0, \alpha_1, \ldots$ at each state. If the α_n are strictly decreasing to 0 as n increase when $0 \le q < 1$, strictly increasing to 1 when $q > 1$, and $\frac{1}{2}$ when $q = 1$, then*

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = \begin{cases} 1 & 0<q<1 \\ \frac{1}{2} & q=1 \\ 0 & q>1 \end{cases} \tag{17}$$

Proof. Let the random variables $X_{t,\alpha}$ follow the Markov chain

$$[1,0,0,\ldots]\begin{bmatrix} \alpha_0 & 1-\alpha_0 & & & 0 \\ & \alpha_1 & 1-\alpha_1 & & \\ & & \alpha_2 & 1-\alpha_2 & \\ & 0 & & \ddots & \ddots \\ & & & & \end{bmatrix}^t \tag{18}$$

We first show that when $0 < q < 1$, $n \ge 0$, $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} \ge (1-\alpha_n)$. See that once $X_{t,\alpha}$ is in position 1, the failure probabilities no longer use α_0, they only depend on $\alpha_1, \alpha_2, \ldots$. For example, if $X_{1,\alpha} = 1$ (we succeed at the first time step) then $X_{t,\alpha}$ from then on follows

$$[0,1,0,\ldots]\begin{bmatrix} \alpha_0 & 1-\alpha_0 & & & 0 \\ & \alpha_1 & 1-\alpha_1 & & \\ & & \alpha_2 & 1-\alpha_2 & \\ & 0 & & \ddots & \ddots \\ & & & & \end{bmatrix}^{t-1} = [1,0,\ldots]\begin{bmatrix} \alpha_1 & 1-\alpha_1 & & \\ & \alpha_2 & 1-\alpha_2 & \\ 0 & & \ddots & \ddots \\ & & & \end{bmatrix}^{t-1}$$

We label this new random variable $X_{t-1,\Sigma\alpha}$, where $\Sigma(\alpha_0, \alpha_1, \ldots) = (\alpha_1, \alpha_2, \ldots)$

Since the likelihood of failure for $X_{t,\alpha}$ is assumed to be strictly decreasing, the likelihood of success $(1-\alpha_n)$ is bounded below by $1-\alpha_0$. This implies that the failure probabilities for $X_{t,\Sigma\alpha}$ are bounded below by $1-(\Sigma\alpha)_0$. Since our random variable is the number of successes in t trials, we get a lower bound on the expectation

$$\mathbb{E}(X_{t,\alpha}) \ge t(1-\alpha_0), \qquad \mathbb{E}(X_{t,\Sigma\alpha}) \ge t(1-\alpha_1) \tag{19}$$

By conditioning the expectation on the result of the first time step, we can get

$$\begin{aligned} \mathbb{E}(X_{t,\alpha}) &= P(X_{1,\alpha}=1)\mathbb{E}(X_{t,\alpha}|X_{1,\alpha}=1) + P(X_{1,\alpha}=0)\mathbb{E}(X_{t,\alpha}|X_{1,\alpha}=0) \\ &= (1-\alpha_0)\mathbb{E}(X_{t-1,\Sigma\alpha}) + \alpha_0\mathbb{E}(X_{t-1,\alpha}) \end{aligned}$$

While it may be possible to solve the recursion, we are only looking for the behavior of the expectation in the large t limit.

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = (1-\alpha_0)\lim_{t\to\infty}\frac{\mathbb{E}(X_{t-1,\Sigma\alpha})}{t-1}\frac{t-1}{t} + \alpha_0\lim_{t\to\infty}\frac{\mathbb{E}(X_{t-1,\alpha})}{t-1}\frac{t-1}{t}$$

By re-indexing and taking the easy limit, we can simplify

$$(1-\alpha_0)\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = (1-\alpha_0)\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t}$$

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = \lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t} \tag{20}$$

As long as $\alpha_0 \neq 1$, the limits are the same. However, we have a better bound on the right side than we do the left due to equation 19

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = \lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} \geq \lim_{t\to\infty}\frac{t(1-\alpha_1)}{t} = 1-\alpha_1$$

So our random variable's normalized expectation is bounded by it's second success probability. But once we have this base case, we then can use it on $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t}$ to get the bound

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t} \geq 1-\alpha_2$$

But because of the equality in equation 20, this is a lower bound for $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t}$ as well.

Specifically, if $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} \geq 1-\alpha_{k-1}$, then through the us of the shifting operator, $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t} \geq 1-\alpha_k$. But equation 20 allows us to use that as a lower bound on the first term,

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} \geq 1-\alpha_k$$

Since this is true for ever $k \geq 1$ and $\lim_{k\to\infty}(1-\alpha_k) = 1$, then we know that when $0 < q < 1$.

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = 1 \tag{21}$$

The other side works as well: if $q > 1$, then the probability of failure is strictly increasing, so the probability of success at the first state is an upper bound for all of the success probabilities. This means that

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = \lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\Sigma\alpha})}{t} \leq \frac{t(1-\alpha_1)}{t}$$

And the same ideas imply that in this case

$$\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} \leq 1-\alpha_n$$

For any $n \geq 1$. Then $\lim_{k\to\infty}(1-\alpha_k) = 0$ implies that $\lim_{t\to\infty}\frac{\mathbb{E}(X_{t,\alpha})}{t} = 0$ in this case.

When $q = 1$, the probability of success at each stage is always $\frac{1}{2}$, so the expectation of the binomial distribution divided by t is $\frac{1}{2}$, and the limit holds through.

□

See that our case, where $\alpha_n = \frac{q^n}{1+q^n}$, the derivative with respect to n is $\frac{q^n\ln(q)}{(q^n+1)^2}$ and thus is strictly decreasing for $q < 1$ and increasing when $q > 1$. Our Markov chain, a q-deformation of the binomial distribution, has a phase transition at $q = 1$.

IV. FUTURE WORK

We would like to examine a number of extensions

1. What if we use higher gradings E^i, or include F? For higher powers of E, like $i=2$, we end up with the transition probabilities

$$P(l \to j) = \begin{cases} \frac{q^{2l}}{1+q^{1+l}(q+q^{-1})+q^{2l}} & j=l \\ \frac{q^{(1+l)}(q+q^{-1})}{1+q^{1+l}(q+q^{-1})+q^{2l}} & j=l+1 \\ \frac{1}{1+q^{1+l}(q+q^{-1})+q^{2l}} & j=l+2 \end{cases}$$

   So there is something to be analyzed there, but will take some more effort.

   On including F, we have

$$\Psi^2(E^1F^1K^l) = q^{-2l}E^1F^1K^{2l} + E^1F^1K^{2l+1} + E^1F^1K^{2l-1} + q^{2l}EFK^{2l} - q^{2l}\frac{K-K^{-1}}{q-q^{-1}}K^{2l}$$

   So that last term does not respect the grading from E,F.

2. Can we categorize the phase transition specifically? This would likely necessitate a more closed form for the expectation, not just how it acts in the limit.

3. Are there other quantum groups where this work can be done?

4. How does our work here use the properties of the quantum group? Not like the Eulerian idempotent and Lyndon words of Diaconis, etc.

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