Last time: · The Double Dual, $(V^*)^*$

①

· For V finite dimensional, $V \longleftrightarrow (V^*)^*$ an isomorphism

$\alpha \longleftrightarrow L_\alpha$ "natural isomorphism"

$L_\alpha : V^* \to \mathbb{F}$, $L_\alpha(f) = f(\alpha)$ no basis needed

S subset of V, S° subspace of V^*, $(S^\circ)^\circ$ subspace of $(V^*)^* \cong V$

$S^{\circ\circ} = \text{span}(S)$ subspace of V $\quad \dim S^\circ + \dim(\text{span} S) = \dim V^* = \dim V$

Ex: $V = \mathbb{R}^2$ $\quad S = \{\begin{pmatrix}1\\0\end{pmatrix}\}$ $\quad$ Find $\underline{S^{\circ\circ}}$ $\quad$ what should it be?

Should be $\text{span}(S) = \{\begin{pmatrix}t\\0\end{pmatrix} : t \in \mathbb{R}\}$ explicit: $S^\circ = \{f \in V^* : f\begin{pmatrix}1\\0\end{pmatrix} = 0\}$

Know $f\begin{pmatrix}x_1\\x_2\end{pmatrix} = a_1 x_1 + a_2 x_2$, so what must $a_1 = $? $\underline{0}$?

Must be of form

(know $\underbrace{\dim W}_{1} + \dim W^\circ = \underbrace{\dim \mathbb{R}^2}_{2}$, so good check)

$S^\circ = \text{span}\{f\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_2\}$ is 1d

$S^{\circ\circ} = \{L \in V^{**} : L(f) = 0 \ \forall f \in S^\circ\}$

L must be of form $L_{\vec\alpha}(f) = f(\vec\alpha)$, $f \in S^\circ \Rightarrow f\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = a_2 \alpha_2 = 0$

What restriction on α? $\underline{\quad}$

$\underline{\alpha_2 = 0}$

So $S^{\circ\circ} = \{L_{\begin{pmatrix}\alpha_1\\0\end{pmatrix}}(f)\} \cong \{\begin{pmatrix}\alpha_1\\0\end{pmatrix} : a_1 \in \mathbb{R}\}$

But $\{\begin{pmatrix}\alpha_1\\0\end{pmatrix}\} = \text{span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$ as expected

· hyperspaces were dimension $n-1$ in $\dim V = n$

what about arbitrary V? $\quad$ lines in $\mathbb{R}^2$, planes in $\mathbb{R}^3$ (surface)

Space that is not V, but "can't" be any bigger.

Def: For V a vector space, a <u>hyperspace</u> W in V is a maximal proper subspace of V

- proper: $W \subset V$, $W \neq V$
- maximal: if E a subspace, $W \subseteq E \subseteq V$, then $W = E$ or $W = V$

So $\alpha \in V \setminus W$ means $\mathrm{span}(W, \alpha) = \underline{\quad} ?\ \underline{V}$

Still have best way to work

Theorem 19: If $f \in V^*$ (a linear functional) that is non-zero $(\exists\, \alpha \in V, f(\alpha) \neq 0)$ then $\mathrm{nullspace}(f)$ is a hyperspace

Conversely, every hyperspace is the nullspace of some functional (not nec. unique)

Ex: $V = \{ \{a_i\}_{i=1}^{\infty} : \sum_{i=1}^{\infty} a_i^2 < \infty \}$ L_2-sequences

Let $f(\{a_i\}) = a_1 \in V^*$

Then $\mathrm{nullspace}(f) = \{ \{a_i\}_{i=1}^{\infty} : a_1 = 0 \}$ is a hyperspace

Proof: Let $f \in V^*$, non-zero, $N_f := \mathrm{nullspace}(f)$ notation. Show N_f is a hyperspace (max & proper)

$\Rightarrow$

Take any $\alpha \in V \setminus N_f$ that is $f(\alpha) \neq 0$ (exists since f non-zero) (means it is proper)

Then show $V = \mathrm{span}(N_f, \alpha)$ (so any 1 more vector gives us everything)

trick from calc: add & subtract since 0

So show $\beta \in V$, $\beta = \gamma + c\alpha$, $\gamma \in N_f$, $c \in F$

$\gamma = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$

Find γ. Since $f\left(\beta - \frac{f(\beta)}{f(\alpha)} \alpha\right) = f(\beta) - \frac{f(\beta)}{f(\alpha)} f(\alpha) = 0$, with $\frac{f(\beta)}{f(\alpha)} \in F$, $f(\alpha) \neq 0$

$\gamma = \beta - c\alpha$

$\boxed{f(\beta) - \underline{\quad} \text{ is } 0?}$

so $\beta - \frac{f(\beta)}{f(\alpha)} \alpha \in N_f$, so $\beta = \left(\beta - \frac{f(\beta)}{f(\alpha)} \alpha\right) + \frac{f(\beta)}{f(\alpha)} \alpha$

③

$\Leftarrow$ Now N a hyperspace, fix $\alpha \in V \setminus N$, so $V = \text{span}(N, \alpha)$ Find $g \in V^*$, nullspace$(g) = N$

Opposite of before

Then $\forall \beta \in V$, $\beta = \gamma + c\alpha$, $\gamma \in N$, $c \in \mathbb{F}$

want $g(\beta) = c$ because if $\beta = \gamma + 0$, then $g(\gamma) = 0$

Want to show γ, c are unique for each β: well defined?

if $\beta \overset{\text{also}}{=} \gamma' + c'\alpha$, then $(c' - c)\alpha = \gamma - \gamma'$

but $\gamma - \gamma' \in N$ as a subspace, so either $\alpha \in N$ or $c' = c$. explain why?

So $g \in V^*$, $g(\beta) := c$ defined above.

Check Linear? $\beta + \delta = (\gamma_1 + c_1\alpha) + (\gamma_2 + c_2\alpha) = (\gamma_1 + \gamma_2) + (c_1 + c_2)\alpha$

$g(\beta + \delta) = c_1 + c_2$

Clearly, nullspace$(g) = N$ $\square$

Proofs in book

Lemma: Let $f, g \in V^*$, then g is a scalar multiple of f iff nullspace of g contains the nullspace of f

$g = cf \iff (f(\alpha) = 0 \Rightarrow g(\alpha) = 0)$

Construct a map that is 0 on V

Theorem 20: Let $g, f_1, \ldots, f_r \in V^*$, null spaces $N, N_1, \ldots, N_r$

g is a linear combination of $f_i \iff N$ contains $\bigcap_{i=1} N_i$

Induct on n (if lin comb exists, easy)

The Transpose

general linear transformations give rise to "new" natural functionals

Let V, W vector spaces over $\mathbb{F}$, $T: V \to W$ linear

$g: W \to \mathbb{F}$, $g \in W^*$

Then $f := g \circ T : V \to W \to \mathbb{F}$ is composition of linear maps, so linear (Theorem 6)

$f \in V^*$

Define function $T^t: W^* \to V^*$

$$T^t(g) = g \circ T$$

T^t is linear: $g_1, g_2 \in W^*$, $c \in \mathbb{F}$, $\alpha \in V$

$$\underbrace{T^t(g_1 + cg_2)}_{\in V^*}(\alpha) = (g_1 + cg_2)(\underbrace{T(\alpha)}_{\in W})$$

$$= g_1(T(\alpha)) + c\, g_2(T(\alpha)) \quad \in \mathbb{F}$$

$$= (\underbrace{T^t g_1}_{\in V^*})(\alpha) + c(\underbrace{T^t g_2}_{\in V^*})(\alpha)$$

Gives us:

Theorem 21: Let V, W be vector spaces over $\mathbb{F}$. For each $T \in \mathcal{L}(V, W)$ there exists a unique $T^t \in \mathcal{L}(W^*, V^*)$ called the transpose such that

$$\forall g \in W^*, \alpha \in V \quad T^t(g)(\alpha) = g(T(\alpha))$$

sometimes adjoint or dual

where else have you heard transpose before?

matrices

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Ex: $\frac{d}{dx} : \mathcal{P} \to \mathcal{P}$ $\quad \varphi \in \mathcal{P}^*$ $\quad \left(\frac{d}{dx}\right)^t(\varphi) \in \mathcal{P}^*$? what

$\left(\frac{d}{dx}\right)^t : \mathcal{P}^* \to \mathcal{P}^*$ $\quad \varphi(p) = p(3)$

(reverse arrow, add star)

$$\left(\frac{d}{dx}^t(\varphi)\right)(p) = \varphi\left(\frac{d}{dx}(p)\right) = p'(3)$$

Ex: $T : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y+z \end{pmatrix}$ $\quad \varphi \in (\mathbb{R}^2)^*$ $\quad \varphi\begin{pmatrix} x \\ y \end{pmatrix} = x+y$ $\quad T^t(\varphi) \in (\mathbb{R}^3)^*$

$T^t : (\mathbb{R}^2)^* \to (\mathbb{R}^3)^*$

$$(T^t(\varphi))\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \varphi\left(T\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \varphi\begin{pmatrix} x+z \\ y+z \end{pmatrix} = x+y+2z$$

Null space and range?

$T : V \to W \quad T^t : W^* \to V^*$

$\text{nullspace}(T^t) = \{ g \in W^* : T^t(g) = 0 \}$ $\quad$ 0 function in V^* $\Rightarrow$ what?

$= \{ g \in W^* : g(T(\alpha)) = 0 \ \forall \alpha \in V \}$ $\quad$ in $\mathbb{F}$

g that sends everything in range $T \to 0$ $\quad$ what is this set?

$= (R(T))^0$ $\quad$ annihilator of range T

If V, W f.d so we can use R-N

$\text{null}(T^t) = \dim(R(T)^0)$ $\quad \dim W^* = \dim W$

$\text{rank}(T^t) = \dim W^* - \text{null}(T^t) = \dim W - \dim(R(T)^0) = \dim(R(T)) = \text{rank}(T)$

R-N $\quad$ Th 16

Range? If $f \in \mathcal{R}(T^t)$ $\exists\, g \in W^*$ s.t. $f = T^t g$

Try null space: $\alpha \in \text{nullspace}(T) =: N$

$$f(\alpha) = T^t g(\alpha) = g(T(\alpha)) = g(0) = 0$$

so $f \in (\text{nullspace}(T))^0 = N^0$

So $\mathcal{R}(T^t) \subseteq N^0$

If F.D.

$$\underset{\substack{\uparrow \\ \text{Thm 16}}}{\dim(N^0) =} \dim V - \underset{\substack{\uparrow \\ R\text{-}N}}{\dim(N) =} \text{rank}(T) \underset{\substack{\uparrow \\ \text{above}}}{=} \text{rank}(T^t)$$

So they equal

This gives:

Theorem 22: $T: V \to W$, then

i) $\text{nullspace}(T^t) = (\mathcal{R}(T))^0$

If V, W f.d.

ii) $\text{rank}(T^t) = \text{rank}(T)$

iii) $\mathcal{R}(T^t) = (\text{nullspace } T)^0$

Remember, if $W^0 = \{0\}$ then $W = V$.

So if T^t injective $(\text{nullspace}(T^t) = \{0\})$

$\mathcal{R}(T) = V$, so T surjective (iff)

In the opposite way

check this

T^t surjective $\Longleftrightarrow$ T injective

How do we represent T^t as a matrix? With respect to dual bases

linear transformation

Theorem 23: Let V, W be finite dimensional v.s. over F

V: basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ dual basis $\mathcal{B}^* = \{f_1, \dots, f_n\}$

W: basis $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ dual basis $\mathcal{B}'^* = \{g_1, \dots, g_m\}$

Setup for matrices

$T: V \to W$, A matrix of T relative to $\mathcal{B}, \mathcal{B}'$

C matrix of T^t relative to $\mathcal{B}^*, \mathcal{B}'^*$

Then $C_{ij} = A_{ji}$

Proof: Since C represents T^t, for $1 \le j \le m$

$$T^t(g_j) = \sum_{r=1}^{n} C_{rj} f_r$$

apply both sides to a basis of V α_k $1 \le k \le n$

$$(T^t(g_j))(\alpha_k) = \sum_{r=1}^{n} C_{rj} f_r(\alpha_k)$$

$\Leftarrow$ use matrix rep first T^t

$$= \sum_{r=1}^{n} C_{rj} \delta_{rk} = C_{kj}$$

Also, $(T^t(g_j))(\alpha_k) = g_j(T(\alpha_k))$ $\Leftarrow$ use matrix rep T

$$= g_j\left(\sum_{r=1}^{m} A_{rk} \beta_r\right)$$

(T represented by A)

$$= \sum_{r=1}^{m} A_{rk}\, g_j(\beta_r)$$

$$= \sum_{r=1}^{m} A_{rk} \delta_{jr} = A_{jk}$$

$\square$

Definition: Let A be an $m \times n$ matrix over $\mathbb{F}$. The <u>transpose</u> of A is A^t, the $n \times m$ matrix $(A^t)_{ij} = A_{ji}$

(Exchange rows and columns) W.r.t. dual bases, A^t rep T^t

$$A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix} \qquad A^t = \begin{pmatrix} 5 & 3 & -4 \\ -7 & 8 & 2 \end{pmatrix}$$

Properties: $(A+C)^t = A^t + C^t \qquad (\lambda A)^t = \lambda A^t \qquad (AC)^t = C^t A^t$

(Check) Know $\text{rank}(T) = \text{rank}(T^t)$ so

Theorem 24: Let A be $m \times n$ matrix over $\mathbb{F}$. Then $\text{row rank}(A) = \text{column rank}(A)$

Proof: $\text{col rank}(A^t) = \text{rank}(T^t) = \text{rank}(T) = \text{column rank}(A)$ $\Leftarrow$ idea

Let $\mathcal{B}$ be standard for $\mathbb{F}^n$, $\mathcal{B}'$ $\mathbb{F}^m$, <u>T be represented by A.</u>

$$T(x_1, \ldots, x_n) = (y_1, \ldots, y_m) \qquad y_i = \sum_{j=1}^{n} A_{ij} x_j$$

$\text{rank}(T) = \text{column rank}(A)$ (linear combinations of columns)

T^t represented by A^t w.r.t. $\mathcal{B}^*, \mathcal{B}'^*$ (Theorem 23)

$\text{rank}(T^t) = \text{column rank}(A^t) = \text{row rank}(A)$ (exchange rows & cols)

$\text{rank}(T^t) = \text{rank}(T)$ (Theorem 22) $\square$

$\text{col rank}(A) = 2 \qquad \text{row rank}(A) = 2$ for example

So if T is represented by A, $\text{rank}(T) = \text{rowrank}(A) = \text{columnrank}(A)$ 9

This is $\underline{\text{rank}(A)}$

Change of basis, formula can be confirmed by the transpose and coordinate function

$T: V \to V$ represented by $\underset{\alpha_i, f_i}{B}$ and $\underset{\beta_i, g_i}{B'}$, $A = [T]_B$ $C = [T]_{B'}$

If $U\alpha_j = \beta_j$ $U_{B \to B'}$ $U^t g_i = f_i$

Then $[T]_{B'} = [U]_B^{-1} [T]_B [U]_B$ (Theorem 14 pg 92)

Remember the dual basis picks out the i^{th} coord: $f_i(\vec{v}) = v_i$

$$T\alpha_j = \sum_{i=1}^{n} A_{ij} \alpha_i$$

$$f_k(T\alpha_j) = \sum_{i=1}^{n} A_{ij} f_k(\alpha_i) = A_{kj}$$

Similarly $g_k(T\beta_j) = C_{kj}$