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Last Time:

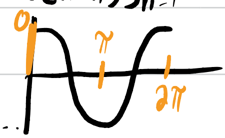
- The definition and terminology of sequences
- Finding the formula for the general term (a_n) of a sequence
- Calculating the limit of a sequence if it exists (using some theorems)

Get Started

a) Find the first 5 terms of the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$

$\cos(1\pi)$, $\cos(2\pi)$

$-1, 1, -1, 1, -1, \ldots$



divergent

b) Find the limit of the sequence $\{(1+\frac{2}{n})^n\}_{n=1}^{\infty} = 1+2, (1+1)^2, (1+\frac{2}{3})^3, \ldots$

$y = (1+\frac{2}{x})^x = f(x)$ $\quad \ln(y) = x\ln(1+\frac{2}{x})$

limit: $\lim\limits_{x\to\infty} x\cdot\ln(1+\frac{2}{x}) \approx (\infty\cdot 0) = \lim\limits_{x\to\infty} \dfrac{\ln(1+\frac{2}{x})}{1/x} \approx \frac{0}{0}$

$$= \lim_{x\to\infty} \frac{\left(\frac{-2/x^2}{1+2/x}\right)}{-1/x^2} = \lim \left(\frac{-2/x^2}{1+2/x}\cdot\frac{-x^2}{1}\right) = 2$$

$\lim\limits_{x\to\infty}\ln(y) = 2 \quad \lim\limits_{x\to\infty} y = e^2$ $\quad$ seq is conv. $\quad$ limit is e^2

c) Find the limit of the sequence $\{\sqrt{n+2}-\sqrt{n}\}_{n=1}^{\infty} = \sqrt{3}-\sqrt{1}, \sqrt{4}-\sqrt{2}, \sqrt{5}-\sqrt{3}, \ldots$

$$\lim_{n\to\infty} \sqrt{n+2}-\sqrt{n}\left(\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}\right)$$

$$\lim_{n\to\infty}\frac{\cancel{n}+2+0-\cancel{n}}{\sqrt{n+2}+\sqrt{n}} = \lim_{n\to\infty}\frac{2}{\sqrt{n+2}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}\left(\sqrt{1+\frac{2}{n}}+1\right)}\cdot 2$$

$= 0$

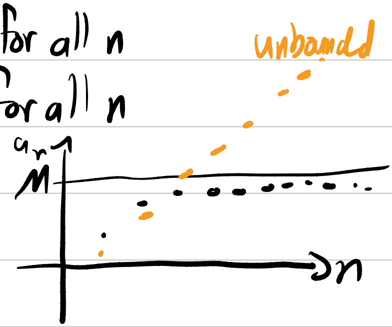
goes to ∞

Monotone Convergence Theorem

Bounded Sequence

Def: A sequence $\{a_n\}$ is <u>bounded above</u> there is a real number M, $a_n \le M$ for all n

<u>bounded below</u> $M \le a_n$ for all n



If a seq. is bounded above & below, we say it is <u>bounded</u>

Otherwise, it is <u>unbounded</u>

Theorem: If a sequence $\{a_n\}$ converges] check
then we know that it is bounded

Note: a sequence can be bounded without converging.

Def: A seq $\{a_n\}$ is <u>increasing</u> if $a_n \le a_{n+1}$ for all n

<u>decreasing</u> if $a_n \ge a_{n+1}$ for all n

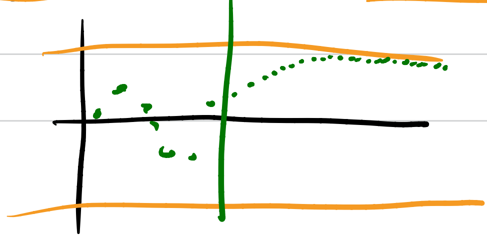
<u>Monotone sequence</u> is increasing or decreasing

Theorem: (Montone Convergence Theorem)

check

If $\{a_n\}$ is a <u>bounded sequence</u> and $\{a_n\}$ is <u>monotone</u> (eventually for all $n \ge N$),

result [then $\{a_n\}$ converges.



Inc: i) $a_n \le a_{n+1}$ (not for $n=1$ or 2, but all n)

ii) $\frac{a_{n+1}}{a_n} \ge 1$

iii) $f(x)$ $f(n) = a_n$ $f'(x) \ge 0$

decreasing is everything flipped

$$\boxed{\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}}$$

if its convergent

biggest # in seq.

Ex: $\left\{\frac{4^n}{n!}\right\}_{n=1}^{\infty} = \frac{4}{1}, \frac{8}{2!}, \frac{32}{3!}, \frac{64}{4!}$

2 $\quad 6$ $\quad \frac{32}{6}$ $\quad \frac{16}{6}$

$4! = 4\cdot 3\cdot 2\cdot 1$

$n! = n\cdot(n-1)\cdot(n-2)\cdots$

$\cdots 3\cdot 2\cdot 1$

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4^n}{n!}\boxed{\frac{4}{(n+1)}} \qquad \frac{4^n}{n!} = a_n$$

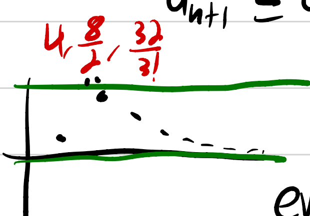
$a_n \boxed{.9} \le a_n \boxed{1}$

$$(n+1)! = (n+1)\cdot \underbrace{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}_{n!}$$

$(n+1)\cdot n!$

$a_{n+1} \le a_n$ when $\frac{4}{n+1} \le 1$ $\quad 4 \le n+1$

$3 \le n$



eventually decreasing, bounded seq.

So it converges

$\{a_2, a_3, a_4, \dots\}$ has the same limit as $\{a_1, a_2, a_3, \dots\}$

$$\lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{4^n}{n!}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$ ← writing a_{n+1} in terms of a_n

simplify

$$\lim_{n\to\infty} \frac{4}{(n+1)} a_n = \lim_{n\to\infty} a_n$$ ← simplify limit

$$\lim_{n\to\infty} \frac{4}{n+1} \cdot \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

$$0 \cdot \lim a_n = \lim a_n \qquad \lim_{n\to\infty} a_n = 0$$

Series

$a_1, a_2, a_3, a_4, \ldots$

↓ add

$a_1 + a_2 + a_3 + a_4 + \cdots$

Throw golf ball 1 meter in the air

each time it bounces half the distance upward

↑ ↓ ↑↓ ↑↓

$$2 + 2\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{8}\right) + \cdots$$

$$2\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots\right)$$

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

$a_n = \frac{1}{2^{n-1}} \quad a_1 = 1, \; a_2 = \frac{1}{2}, \; a_3 = \frac{1}{4}$

$S_1 = 1$ meter

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4}$

$S_4 = S_3 + \frac{1}{8} = \frac{15}{8}$

$S_n \quad n \to \infty$

1# $\quad$ 1# (sum of 2) $\quad$ 2#

$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3$

$S_1 = \sum_{k=1}^{1} a_k \qquad S_2 = \sum_{k=1}^{2} a_k \qquad S_3 = \sum_{k=1}^{3} a_k$

$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ 1#

$n \to \infty$

$\sum_{k=1}^{\infty} a_k \qquad \lim_{n \to \infty} S_n$

$\{S_1, S_2, S_3, S_4, \ldots\}$

$S_5 = 1.9375 \qquad S_{20} = 1.999998$

$\lim_{n \to \infty} S_n = 2$

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = 2$

Def: An infinite series is an infinit sum of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The k^{th} partial sum is $\sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k$ is a # we can find

We can form a sequence (a list) $\{S_1, S_2, S_3, \ldots\} = \{S_k\}_{k=1}^{\infty}$

If the sequence of partial sums is convergent (to $S = \lim_{k\to\infty} S_k$)
The infinite series converges.

$$S = \sum_{n=1}^{\infty} a_n$$

If the sequence of partial sums diverges
the series diverges

Ex Using sigma (Σ) notation

a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ $\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\}$ $\left(\frac{1}{3}\right)^{n-1}$

Series

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \cdots$$

+1 -1

$$= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \cdots$$

b) $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$ $\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\}$

$a_n \quad (-1)^n \cdot \frac{1}{n}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

c) $18\left(\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\cdots\right)$

$a_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$$18 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

d) $0.777\ldots = 0.\overline{7} = \cancel{0} + \frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\cdots$

$a_n = \frac{7}{(10)^n}$

$$\sum_{n=1}^{\infty} \frac{7}{(10)^n}$$

later

Ex: a) $S_n = \sum_{k=1}^{n} a_k = a_1+a_2+\cdots+a_n = \frac{2n}{3n+5}$ $\{S_1, S_2, \ldots\}$ $\{S_n\}_{n=1}^{\infty}$

$$\boxed{\sum_{k=1}^{\infty} a_k} = S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} \frac{2n}{3n+5} = \boxed{\frac{2}{3}}$$

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

1) Find the partial sums S_n (write them out)

2) Find a formula for them S_n

3) Take $\lim_{n\to\infty} S_n$

$S_1 = a_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$

1 $S_2 = a_1+a_2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$

$S_3 = a_1+a_2+a_3 = S_2+a_3 = \frac{2}{3}+\frac{1}{12} = \frac{3}{4}$

$S_4 = S_3 + a_4 = \frac{3}{4} + \frac{1}{20} = \frac{15}{20} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$

2 [n^{th} partial sum $S_n = \frac{n}{n+1}$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = \underline{1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = S = \underline{1}$$

c) $\sum_{n=1}^{\infty} (-1)^n$

$S_1 = a_1 = -1$

$S_2 = -1 + 1 = 0$

$S_3 = S_2 + a_3 = 0 - 1 = -1$

$S_4 = S_3 + a_4 = -1 + 1 = 0$

$\{S_n\} = \{-1, 0, -1, 0, \dots\}$ diverges

$\sum_{n=1}^{\infty} (-1)^n$ diverges (DNE)

d) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$S_1 = \frac{1}{2}$ $\frac{1}{2}$

$S_2 = S_1 + a_2 = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$

$S_3 = \frac{7}{6} + a_3 = \frac{7}{6} + \frac{3}{4} = \frac{14+6}{12} = \frac{20}{12} = \frac{4}{3} > 1$

$S_4 = \frac{4}{3} + a_4 = \frac{4}{3} + \frac{4}{5} = \frac{20+12}{15} = \frac{32}{15} > 2$

This is unbounded

$\{S_n\}_{n=1}^{\infty}$ is unbounded so it cannot converge.

So $\{S_n\}$ diverges $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges

$S_1 = \frac{1}{2}$ $\quad S_2 = \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right)$

$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right)$

$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > 4\left(\frac{1}{2}\right)$

$S_n > n \cdot \left(\frac{1}{2}\right)$

$\lim_{n\to\infty} n \cdot \frac{1}{2} = \infty$

$\lim_{n\to\infty} S_n = \infty$

S_n is unbounded $\quad M$

$n \cdot \frac{1}{2} < S_n \leq M$

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\quad S_1 = \frac{2}{1}$ $\quad S_2 = \frac{2}{1} + \frac{3}{2} > 1 + 1$

$S_3 = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} > 1 + 1 + 1$ $\quad S_k > n \cdot 1$

seq. of partials sums ~~are~~ unbounded $\Rightarrow$ diverges

series also diverges

Theorem: If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ convergent series] check

i) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges $\quad = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

ii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum a_n - \sum b_n$ $\quad$ converges

iii) Real # c, $\quad \sum_{n=1}^{\infty} c a_n = c a_1 + c \cdot a_2 + c \cdot a_3 + \cdots$

$= c \sum_{n=1}^{\infty} a_n$

Ex: a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right)$

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}$$

$$3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{-1}\left(\frac{1}{2}\right)^{n-1}$$

$$3\cdot 1+\left(\frac{1}{2}\right)^{-1}(2)$$

$$3+2(2)=3+4=7$$

Assume/have shown

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}=2$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=S=1$$

$$\left(\frac{1}{2}\right)^{n-2}=\left(\frac{1}{2}\right)^{n-1}\cdot\left(\frac{1}{2}\right)^{-1}$$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}=5\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$

$$=5\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=5\cdot 2=10$$

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}=2$$

Harmonic Series (from music)

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

Diverges

S_k are unbounded

seq. of partial sums diverges

so the series does as well

Geometric Series

$$a + a\cdot r + a\cdot r^2 + a\cdot r^3 + \cdots = \sum_{n=1}^{\infty} a\cdot r^{n-1}$$

$a \to$ initial term

$r \to$ ratio

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$a = 2 \qquad r = \frac{1}{2}$ $\qquad a_n = a\cdot r^{n-1}$

different

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{a}\cdot r^{(n+1)-1}}{\cancel{a}\cdot r^{n-1}} = \frac{r^n}{r^{n-1}} = r^{n-(n-1)} = r$$

$$S_n = a\cdot\frac{(1-r^n)}{1-r} \qquad r \neq 1$$

$r \neq 1 \qquad r = -1$

$$\lim_{n\to\infty} S_n = a\cdot\lim_{n\to\infty}\frac{1-r^n}{1-r}$$

r^n diverges if $|r| > 1$

r^n converges if $|r| < 1$

either diverges or

$$S = a\cdot\frac{1}{1-r} \qquad \text{as long as } |r| < 1$$

When $|r| < 1$ (and for any a)

The geometric series $\sum_{n=1}^{\infty} a\cdot r^{n-1} = \frac{a}{1-r}$

converges

If $|r| \geq 1$ it diverges

$$\sum_{n=1}^{\infty} 2\cdot\left(\frac{1}{2}\right)^{n-1} = \frac{2}{1-1/2} = \frac{2}{1/2} = 2\cdot\frac{2}{1} = 4$$

Manipulate: $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}$

$$a \quad + ar \quad + a \cdot r^2 + \cdots$$

i) write out the terms: $\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots$

$$a = \left(\frac{2}{3}\right)^2 = \frac{4}{9} \qquad r = \frac{2}{3}$$

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}$$

$$|r| = \left|\frac{2}{3}\right| < 1 \qquad S = \frac{a}{1-r} = \frac{4/9}{1-2/3}$$

converges

$$\frac{4/9}{1/3} = \frac{4}{9} \cdot 3 = \frac{4}{3}$$

ii) shift index $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}$ $\quad +1 \quad -1$

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{(n-1)+2} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1} = \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{3}\right)^2}_{a} \underbrace{\left(\frac{2}{3}\right)^{n-1}}_{r}$$

Ex: a) $\sum_{k=1}^{\infty} \frac{(-3)^{k+1}}{4^{k-1}} = \sum_{k=1}^{\infty} \underbrace{(-3)^2}_{a} \underbrace{\frac{(-3)^{k-1}}{4^{k-1}}}_{r}$

$$\frac{(-3)^{k-1}}{4^{k-1}} = \left(\frac{-3}{4}\right)^{k-1}$$

$$r = \frac{-3}{4} \qquad a = (-3)^2 = 9$$

$$|r| = \left|\frac{-3}{4}\right| = \frac{3}{4} < 1 \qquad S = \frac{a}{1-r}$$

$$r = \frac{-3}{4} \qquad = \frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{9 \cdot 4}{7} = \frac{36}{7}$$

Note $r < 0$
$1 - r$
$1 + \_$

b) $\sum_{n=1}^{\infty} e^{2n} = \underbrace{e^2}_{a} + e^4 + e^6$

$$r = e^2 \qquad \cancel{|e^2| < 1}$$

series diverges DNE

$a=1$

c) $\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^{n-1}$

$\left(-\frac{2}{5}\right)^{1-1} = \boxed{1} + \left(-\frac{2}{5}\right) + \left(-\frac{2}{5}\right)^2 + \cdots$

$S = \frac{a}{1-r}$ $\quad |r| = \left|-\frac{2}{5}\right| < 1$

$= \frac{1}{1-(-2/5)} = \frac{1}{1+2/5} = \frac{1}{7/5} = \frac{5}{7}$