The Topology of Magmas

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Definition (Magma)

A magma (or binar or, classically, groupoid) is an algebraic structure \((S, f)\) consisting of an underlying set \(S\) and a single binary operation \(f : S^2 \rightarrow S\).
Operation Digraphs

Definition (Operation digraph)
Let \( f : S \to S \) be a unary operation. The operation digraph (or functional digraph) of \( f \), written \( G_f \), is given by \( G_f = G(S, E) \) where

\[
E = \{ (s, f(s)) \mid s \in S \}.
\]

Definition (Operation digraph for a binary operation)
Let \( f : S^2 \to S \) be a binary operation and let \( s \in S \). The left operation digraph of \( s \) under \( f \), written \( G_{f_s}^L \), is the operation digraph of \( f_s^L : S \to S \) where \( f_s^L(x) := f(s, x) \) for \( x \in S \). The right operation digraph of \( s \) under \( f \), written \( G_{f_s}^R \), is defined analogously.
Example: Operation Digraphs from $\mathbb{Z}/3\mathbb{Z}$
Previous Work in...

- Semigroup theory
- Dynamics and number theory
- Cayley graphs
- Graph theory
- Universal algebra (unary algebras)
Definition (Adjacency matrix)

Let $G(V, E)$ be a digraph, let $|V| = n$, and fix an order on the vertex set $V$. The adjacency matrix $A$ for $G$ under the given order on $V$ is the $n \times n$ matrix whose $ij$-entry is 1 if there is an edge in $G$ from $v_i$ to $v_j$ and 0 otherwise.

We write $A^L_{fs}$ to indicate the adjacency matrix of $G^L_{fs}$ and similarly write $A^R_{fs}$ to indicate the adjacency matrix of $G^R_{fs}$. 
Example: Operation Matrices from $\mathbb{Z}/3\mathbb{Z}$

\[
A_{+0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_{+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{+2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

\[
A_{\times0} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{\times1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_{\times2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]
Example: Operation Matrices from $\mathbb{Z}/3\mathbb{Z}$

Write $s_i$ to indicate $i$ viewed as an element of $\mathbb{Z}/3\mathbb{Z}$. Multiplying a vector by the adjacency matrix of an operation digraph corresponds to applying the corresponding function to the corresponding element.

\[
s_2A_{+1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = [1 \ 0 \ 0] = s_0
\]

\[
s_1A_{+2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [1 \ 0 \ 0] = s_0
\]
Graph Treks

Theorem

Let $A$ be the adjacency matrix for $G$ with a given vertex ordering. Then $(A^k)_{ij}$ for $k \in \mathbb{N}$ is the number walks of length $k$ from $v_i$ to $v_j$ in $G$.

It is natural to consider the significance of the product of the adjacency matrices of two or more different graphs on the same set of vertices.
Graph Treks

**Definition (Trek)**

Let \((G_1, G_2, \ldots, G_k)\) be a tuple of graphs on a common set of vertices \(V\). A **trek** (or \((v_i, v_j)\)-trek) on \((G_1, G_2, \ldots, G_k)\) is an ordered list of vertices and edges \(v_i, e_1, \ldots, e_k, v_j\) where \(e_t \in E(G_t)\) is an edge joining the vertices before and after it in the list.

**Theorem (A. 2015)**

Let \((G_1, G_2, \ldots, G_k)\) be a tuple of graphs on a set of vertices \(V\) under a given vertex ordering and let \(A_1, A_2, \ldots, A_k\) be the corresponding adjacency matrices. Then \((A_1A_2\cdots A_k)_{ij}\) is the number of treks on \((G_1, G_2, \ldots, G_k)\) of length \(k\) from \(v_i\) to \(v_j\).
Multiplying operation matrices corresponds to function composition:

\[
A \times_2 A_{+1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

This also corresponds to looking at those treks which consist of a step on \( G \times_2 \) followed by a step on \( G_{+1} \).
Theorem (A. 2015)

Let $S$ be an ordered finite set of elements and let $\{f_p\}_{p \in P}$ where $f_p : S \to S$ be an indexed collection of functions. Let $G_p = G(S, E_p)$ be the operation digraph for $f_p$ and let $A_p$ be the adjacency matrix for $G_p$ under the given ordering for $S$. If $Q = \{q_n\}_{n=1}^k$ is a finite sequence of $k$ elements of $P$ and $y = s_j$ is a fixed element of $S$ we have that the number of $x \in S$ for which $f^Q(x) = y$ is exactly $\sum_{i=1}^{|S|} \left( \prod_{n=1}^k (A_{q_n}) \right)_{ij}$. 


Theorem (Sylvester’s Rank Inequality)

Let $U$, $V$, and $W$ be finite-dimensional vector spaces, let $A$ be a linear transformation from $U$ to $V$ and let $B$ be a linear transformation from $V$ to $W$. Then

$$\text{rank } BA \geq \text{rank } A + \text{rank } B - \dim V.$$ 

By induction we see that for a finite collection of linear transformations $\{A_i : V \to V\}_{i \in I}$ we have

$$\text{rank } \prod_{i \in I} A_i \geq \left( \sum_{i \in I} \text{rank } A_i \right) - (|I| - 1) \dim V.$$
Example: An Equation over $\mathbb{Z}/4\mathbb{Z}$

$((3(x + 2))^3)((3(x+2))^3) = y$

Let $f_1(x) = x + 2$, $f_2(x) = 3x$, $f_3(x) = x^3$, and $f_4(x) = x^x$. Note that the equation under consideration can be rewritten as $f^Q(x) = y$, where $Q$ is the sequence $(1, 2, 3, 4)$.

$A_1 = A_{+2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$A_2 = A_{\times3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$A_3 = A^{R}_{\wedge3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A_4 = A^{R}_{\uparrow2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Example: An Equation over $\mathbb{Z}/4\mathbb{Z}$

\[
((3(x + 2))^3)((3(x+2))^3) = y
\]

Since rank $A_{+2} = \text{rank } A_{x3} = 4$ and rank $A_{\wedge3} = \text{rank } A_{\uparrow2} = 3$, we have that

\[
\text{rank } \prod_{n=1}^{4} A_n \geq \left( \sum_{n=1}^{4} \text{rank } A_n \right) - (|I| - 1)|S|
\]

\[
= (4 + 4 + 3 + 3) - (4 - 1)4
\]

\[
= 2.
\]
Sylvester’s Inequality for Functions

Proposition (Sylvester’s inequality for functions)

Let $X$, $Y$, and $Z$ be finite sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

$$|(g \circ f)(X)| \geq |f(X)| + |g(Y)| - |Y|.$$
Definition (Operation hypergraph)

Let \( f : S^2 \to S \) be a binary operation. The operation hypergraph of \( f \), written \( G_f \), is given by \( G_f = G(S, E) \) where

\[
E = \{(s_i, s_j, f(s_i, s_j)) \mid s_i, s_j \in S\}.
\]
Definition (Adjacency tensor)

Let $G(V, E)$ be a 3-uniform hypergraph, let $|V| = n$, and fix an order on the vertex set $V$. The \textit{adjacency tensor} $A$ for $G$ under the given order on $V$ is the $n \times n \times n$ hypermatrix whose $ijk$-entry is 1 if $(v_i, v_j, v_k)$ is an edge in $G$ and 0 otherwise.

Recall that given such a tensor we can obtain a bilinear map $A_f: \mathbb{C}^S \times \mathbb{C}^S \to \mathbb{C}^S$ where given $x_1 = (a_s s)_{s \in S}$ and $x_2 = (b_s s)_{s \in S}$ from $\mathbb{R}^S$ we define

$$A_f(x_1, x_2) := \sum_{s_i, s_j, s_k \in S} a_{s_i} b_{s_j} (A_f)_{ijk} s_k = \sum_{s_i, s_j \in S} a_{s_i} b_{s_j} f(s_i, s_j).$$
There are many ways to compose binary operations. Let $f, g : S^2 \rightarrow S$.

$$(x, y, z) \mapsto g(f(x, y), z)$$

$$(x, y, z) \mapsto f(f(x, x), g(x, f(x, f(y, z)))).$$
We return to our $2x + 1 = y$ example.
Definition ($\mu, \Sigma$-odyssey)

Let $X$ and $Y$ be sets of variables and take $\Sigma$ to be a collection of pairs of the form $(e, E)$ where $E = E_i$ for some $i \in I$ and $e \in (X \cup Y)^{\rho(i)}$. If there exist evaluation maps $\mu: X \rightarrow S$ (the endpoint evaluation map) and $\nu: Y \rightarrow S$ (the intermediate point evaluation map) such that for each $(e, E) \in \Sigma$ we have that $(\mu \circ \nu)(e) \in E$ then we say that the collection of edges $\mathcal{O} = (\mu \circ \nu)(e)$ is a $\Sigma$-odyssey on the $G_i$. We say that $X$ is the set of end variables, $Y$ is the set of intermediate variables, $\mu(X)$ is the set of endpoints, $\nu(Y)$ is the set of intermediate points, $\Sigma$ is the odyssey type, and $|\Sigma|$ is the length of the odyssey. We call a $\Sigma$-odyssey $\mathcal{O}$ a $\mu, \Sigma$-odyssey if $\mu: X \rightarrow S$ is the endpoint evaluation map of $\mathcal{O}$ for some fixed $\mu$. 
Hypergraph Odysseys

$y = ax + b$
$t = ax$

End variables: $X = \{x, y, a, b\}$
Intermediate variable: $Y = \{t\}$
Odyssey type: $\Sigma = \{((a, x, t), G_x), ((t, b, y), G_+)\}$
Let $\varphi$ denote the logical formula

$$\varphi(a, b, x, y) := (\exists t \in \mathbb{Z}/3\mathbb{Z})((a, x, t) \in G \times (a, x, t) \in G \times G_+).$$

Let $A$ and $B$ be arbitrary rank 3 tensors over $\mathbb{C}$. Define

$$(\varphi AB)_{ijkl} := \sum_{t \in \{0, 1, 2\}} A_{ikt} B_{tjl},$$

which is the generalized matrix product of $A$ and $B$ corresponding to the logical formula $\varphi$. By simple definition-chasing one finds that $\varphi G \times G_+$ is the adjacency tensor for the composite operation

$$(a, b, x) \mapsto ax + b.$$
Definition (Operation graph)

Let \( f : S \to S \) be a unary operation. The operation graph of \( f \), written \( \bar{G}_f \), is the simple graph \( G(V, E) \) which is constructed as follows. For each edge \( e = (s, f(s)) \) in \( G_f \) define

\[
\sigma(e) := \begin{cases} 
\{(s, u_e), (u_e, v_e), (v_e, s)\} & \text{when } f(s) = s \\
\{(s, u_e), (u_e, f(s))\} & \text{when } f^2(s) = s \text{ and } f(s) \neq s \\
\{e\} & \text{otherwise}
\end{cases}
\]

where \( u_e \) and \( v_e \) are new vertices unique to the edge \( e \). Take \( E = \bigcup_{e \in E(G_f)} \sigma(e) \) and let \( V \) be the union of \( S \) and all the \( u_e \) and \( v_e \) generated by applying \( \sigma \) to edges \( e \in E(G_f) \).
Embedding Dimension

\[ s = f(s) \]

\[ \sigma \]

\[ s = f(f(s)) \]

\[ s = f(s) \]

\[ \sigma \]

\[ s = f(f(s)) \]
Embedding Dimension

**Theorem**

*Every operation graph is planar.*

**Theorem**

*Let $H$ be a subdivision of a simple graph $H'$ with $n$ vertices, each of degree at least $k + 1$ for $k \geq 2$. The graph $H$ cannot appear as a subgraph of any operation graph if $k > \frac{n-1}{2}$.*
Definition (Operation complex)

Let \( f : S^2 \to S \) be a binary operation. The operation complex of \( f \), written \( \tilde{G}_f \), is the simplicial complex whose 2-faces are the edges of the hypergraph \( G(V, E) \), which is constructed as follows. Write \((a, b, c, d)_2\) to indicate the set of all 2-faces of the simplex with vertices \( a, b, c, \) and \( d \). For each edge \( e = (s_i, s_j, f(s_i, s_j)) \) in \( G_f \) define

\[
\sigma(e) := \begin{cases} 
(s_i, u_e, v_e, w_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 1 \\
(s_i, s_j, u_e, v_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 2 \\
(s_i, s_j, s_k, u_e)_2 & \text{when } |\{s_i, s_j, f(s_i, s_j)\}| = 3 \text{ and } \tau e \in f \text{ for some nonidentity permutation } \tau \\
\{e\} & \text{otherwise}
\end{cases}
\]

where \( u_e, v_e, \) and \( w_e \) are new vertices unique to the edge \( e \). Take \( E = \bigcup_{e \in E(G_f)} \sigma(e) \) and let \( V \) be the union of \( S \) and all the \( u_e, v_e, \) and \( w_e \) generated by applying \( \sigma \) to edges \( e \in E(G_f) \).
Given any magma \((S, f)\) we then know that \(\tilde{G}_f\) embeds into \(\mathbb{R}^k\) but not \(\mathbb{R}^{k-1}\) for some \(k \in \{3, 4, 5\}\).

**Definition (Embedding dimension)**

Let \((S, f)\) be a magma with operation complex \(\tilde{G}_f\). We refer to the minimal \(k\) such that the complex \(\tilde{G}_f\) embeds into \(\mathbb{R}^k\) as the *embedding dimension* of the magma \((S, f)\).

The situation here is more complex than for unary operations.
Embedding Dimension

Let \((S, f)\) be a magma such that for every \(x, y \in S, x \neq y\), we have that either \(f(x, y) = x\) or \(f(x, y) = y\). Every edge \(e \in G_f\) then contains at most 2 vertices which belong to \(S\). We can embed \(\bar{G}_f\) into \(\mathbb{R}^3\) without self-intersections. There are also magmas of embedding dimension 3 without this property. Consider \((\mathbb{Z}_3, +)\).
Consider the triangulation $K_{h12}$ of the Klein bottle.
We orient faces to obtain a partial operation table.

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This “forbidden substructure” cannot appear in any magma with embedding dimension 3.
If a magma has embedding dimension $n$ then clearly every submagma has embedding dimension at most $n$. How does embedding dimension behave under taking products or homomorphic images of magmas?

If the class “magmas of embedding dimension at most $n$” is closed under taking homomorphic images, submagmas, and products we would have an equational class (Birkhoff’s Variety Theorem).
Theorem

Let $f : S \to S$ be a function on a set $S$ of size $n$. Let $m(j)$ denote the number of $j$-cycles under $f$ and let $Z_j$ denote the multiset which consists of $m(j)$ copies of each $j^{th}$ root of unity. The nonzero part of the spectrum of $A_f$ is the multiset union $\bigcup_j Z_j$. 
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