Orientable smooth manifolds are essentially quasigroups

Charlotte Aten (joint work with Semin Yoo)

University of Rochester

2022 January 25
Introduction

- In the mid-2010s Herman and Pakianathan introduced a functorial construction of closed surfaces from noncommutative finite groups.
- Semin Yoo and I decided to produce an $n$-dimensional generalization.
- The two main challenges in doing this were finding an appropriate analogue of noncommutative groups and in desingularizing the $n$-dimensional pseudomanifolds which arose at the first stage of our construction.
- Ultimately we found that every orientable triangulable manifold could be manufactured in the manner we described.
Talk outline

- Herman and Pakianathan’s construction
- Quasigroups
- The first functor: Open serenation
- The second functor: Serenation
- The Evans Conjecture and Latin cubes
Consider the quaternion group $G$ of order 8 whose universe is $G := \{\pm 1, \pm i, \pm j, \pm k\}$.

We begin by picking out all the pairs of elements $(x, y) \in G^2$ so that $xy \neq yx$. We call this collection $\text{NCT}(G)$.

We define $\text{In}(G)$ to be all the elements of $G$ which are entries in some pair $(x, y) \in \text{NCT}(G)$.

Similarly, $\text{Out}(G)$ is defined to be all the members of $G$ of the form $xy$ where $(x, y) \in \text{NCT}(G)$. 
Herman and Pakianathan’s construction

In this case we have

\[
\mathrm{NCT}(G) = \left\{ (\pm u, \pm v) \mid \{u, v\} \in \binom{\{i, j, k\}}{2} \right\}
\]

so

\[
\ln(G) = \{\pm i, \pm j, \pm k\}
\]

and

\[
\out(G) = \{\pm i, \pm j, \pm k\}.
\]

From this data we form a simplicial complex (actually a 2-pseudomanifold) whose facets are of the form \(\{x, y, xy\}\) where \((x, y) \in \mathrm{NCT}(G)\).
Herman and Pakianathan’s construction

One «sheet» of this complex is pictured below.
Herman and Pakianathan’s construction

- The three 4-cycles

\((i, j, -i, -j), (i, k, -i, -k), \) and \((j, k, -j, -k)\).

each carry an octohedron.
Herman and Pakianathan’s construction

- This simplicial complex, which we call $\text{Sim}(G)$ and Herman and Pakianathan called $X(Q_8)$, consists of three 2-spheres, each pair of which is glued at two points.
- Deleting these points to disjointize the spheres and filling the resulting holes yields the manifold we call $\text{Ser}(G)$ and Herman and Pakianathan called $Y(Q_8)$.
- In this case $\text{Ser}(G)$ is the disjoint union of three 2-spheres.
Quasigroups

Definition (Quasigroup)

A (binary) quasigroup is a magma $A := (A, f: A^2 \to A)$ such that if any two of the variables $x$, $y$, and $z$ are fixed the equation

$$f(x, y) = z$$

has a unique solution.

- That is, a quasigroup is a magma whose Cayley table is a Latin square, where each entry occurs once in each row and each column.
- All groups are quasigroups, but quasigroups need not have identities or be associative.
Quasigroups

- The midpoint operation
  \[ f(x, y) := \frac{1}{2}(x + y) \]
  is a quasigroup operation on \( \mathbb{R}^n \).
- The magma \( (\mathbb{Z}, -) \) is a quasigroup.
Quasigroups

Definition (Quasigroup)

A (binary) quasigroup is an algebra $\mathbf{A} := (A, f, g_1, g_2)$ where for all $x_1, x_2, y \in A$ we have

$$f(g_1(x_2, y), x_2) = y,$$

$$f(x_1, g_2(x_1, y)) = y,$$

$$g_1(x_2, f(x_1, x_2)) = x_1,$$

and

$$g_2(x_1, f(x_1, x_2)) = x_2.$$ 

We think of $g_1(x, y)$ as the division of $y$ by $x$ in the second coordinate.
The preceding definition shows that the class $\text{Quas}_2$ of all binary quasigroups can be defined by universally-quantified equations, or *identities*.

This means that $\text{Quas}_2$ is a variety of algebras in the sense of universal algebra, and hence forms a category $\text{Quas}_2$ which is closed under taking quotients, subalgebras, and products.

Note that Herman and Pakianathan’s construction works with noncommutative quasigroups just as well as with groups.

We would then like an $n$-ary version of a quasigroup for our $n$-dimensional generalization.
Definition (Quasigroup)

An $n$-quasigroup is an $n$-magma $A := (A, f : A^n \to A)$ such that if any $n - 1$ of the variables $x_1, \ldots, x_n, y$ are fixed the equation

$$f(x_1, \ldots, x_n) = y$$

has a unique solution.

- That is, an $n$-quasigroup is an $n$-magma whose Cayley table is a Latin $n$-cube.
- All $n$-ary groups are quasigroups, but quasigroups need not be associative.
Quasigroups

Given any group $G$ the $n$-ary multiplication

$$f(x_1, \ldots, x_n) := x_1 \cdots x_n$$

is a quasigroup operation on $G$. 
Quasigroups

Definition (Quasigroup)

An \textit{n-quasigroup} is an algebra

\[ A := (A, f, g_1, \ldots, g_n) \]

where for all \( x_1, \ldots, x_n, y \in A \) and each \( i \in \{1, 2, \ldots, n\} \) we have

\[ f(x_1, \ldots, x_{i-1}, g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y), x_{i+1}, \ldots, x_n) = y \]

and

\[ g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n)) = x_i. \]

- We think of \( g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y) \) as the division of \( y \) simultaneously by \( x_j \) in the \( j^{th} \) coordinate for each \( j \neq i \).
Quasigroups

- We say that an \(n\)-quasigroup \(A\) is *commutative* when for all \(x_1, \ldots, x_n \in A\) and all \(\sigma \in \text{Perm}_n\) we have
  \[ f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \]

- We say that an \(n\)-quasigroup \(A\) is *alternating* when for all \(x_1, \ldots, x_n \in A\) and all \(\sigma \in \text{Alt}_n\) we have
  \[ f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \]

- Our “correct” analogue of the variety of groups will be the variety \(AQ_n\) of alternating \(n\)-ary quasigroups.
Quasigroups

- There are nontrivial members of AQ$_n$ for each $n$, but the easiest examples are either commutative (take the $n$-ary multiplication for an abelian group) or infinite (the free alternating quasigroups, which we need later but are too much right now).
- We tediously found the following example by hand:
Quasigroups

Take \( S := (\mathbb{Z}/5\mathbb{Z})^3 \) and define \( h: \mathbb{Z}/5\mathbb{Z} \times \text{Alt}_3 \rightarrow \text{Perm}_S \) by

\[
(h(k, \sigma))(x_1, x_2, x_3) := (x_{\sigma(1)} + k, x_{\sigma(2)} + k, x_{\sigma(3)} + k).
\]

There are 7 members of \( \text{Orb}(h) \). One system of orbit representatives is:

\[
\{000, 011, 022, 012, 021, 013, 031\}.
\]
Quasigroups

Let \( A := \mathbb{Z}/5\mathbb{Z} \) and define a ternary operation \( f: A^3 \rightarrow A \) so that

\[
f((h(k, \sigma))(x_1, x_2, x_3)) = f(x_1, x_2, x_3) + k
\]

and \( f \) is defined on the above set of orbit representatives as follows.

<table>
<thead>
<tr>
<th>xyz</th>
<th>( f(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0</td>
</tr>
<tr>
<td>011</td>
<td>0</td>
</tr>
<tr>
<td>022</td>
<td>0</td>
</tr>
<tr>
<td>012</td>
<td>3</td>
</tr>
<tr>
<td>021</td>
<td>4</td>
</tr>
<tr>
<td>013</td>
<td>4</td>
</tr>
<tr>
<td>031</td>
<td>2</td>
</tr>
</tbody>
</table>
Quasigroups

- By taking products of $\mathbf{A} := (A, f)$ this gives us infinitely many finite, noncommutative, alternating ternary quasigroups, but we only have one basic example.

- We reached out to Jonathan Smith to see if anyone had studied the varieties of alternating $n$-quasigroups before, but it seemed that no one had.

- He did, however, give us an example which we generalized into an *alternating product* construction which takes an $n$-ary commutative quasigroup and an $(n + 1)$-ary commutative quasigroup and yields an $n$-ary alternating quasigroup which is typically not commutative.
Quasigroups

**Definition (Alternating map)**

Given sets $A$ and $B$ we say that a function $\alpha: A^n \rightarrow B$ is an \textit{n-ary alternating map} from $A$ to $B$ when for each $\sigma \in \text{Alt}_n$ and each $a \in A^n$ we have that

$$\alpha(a) = \alpha(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

- Note that the determinant is an alternating $n$-ary map from $\mathbb{F}^n$ to $\mathbb{F}$ for any field $\mathbb{F}$. 
Quasigroups

**Definition (Alternating product)**

Given an $n$-ary commutative quasigroup $U := (U, g)$, an $(n + 1)$-ary commutative quasigroup $V := (V, h)$, and an $n$-ary alternating map $\alpha: A^n \to B$ the *alternating product* of $U$ and $V$ with alternating map $\alpha$ is the alternating $n$-quasigroup

$$U \boxtimes_{\alpha} V := (U \times V, g \boxtimes_{\alpha} h: (U \times V)^n \to U \times V)$$

where for $(u_1, v_1), \ldots, (u_n, v_n) \in U \times V$ we define

$$(g \boxtimes_{\alpha} h)((u_1, v_1), \ldots, (u_n, v_n)) := (g(u), h(\alpha(u), v_1, \ldots, v_n))$$

where $u := (u_1, \ldots, u_n)$. 
Quasigroups

- The variety of $n$-quasigroups (not necessarily alternating) is congruence permutable, and hence congruence modular.

- Note the similarity between the alternating product $U \boxtimes_\alpha V$ and the decomposition decomposition of an algebra $A$ in a congruence modular variety as $Q \otimes^T B$ where $Q$ is Abelian and $B := A/\zeta_A$.

- Note also the similarity between this construction and the factor set construction of group extensions with an abelian kernel.
Quasigroups

**Definition (Commuting tuple)**

Given \( A := (A, f) \in AQ_n \) we say that \( a \in A^n \) *commutes* (or is a *commuting tuple*) in \( A \) when we have for each \( \sigma \in \text{Perm}_n \) that

\[
f(a) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).
\]

**Definition (Set of noncommuting tuples)**

Given \( A := (A, f) \in AQ_n \) we define the *noncommuting tuples* \( \text{NCT}(A) \) of \( A \) by

\[
\text{NCT}(A) := \{ a \in A^n \mid a \text{ does not commute in } A \}.
\]
Quasigroups

Definition (NC homomorphism)

We say that a homomorphism \( h: A_1 \to A_2 \) of alternating quasigroups is an \textit{NC homomorphism} (or a \textit{noncommuting homomorphism}) when for each \( a \in \text{NCT}(A_1) \) we have that

\[
h(a) = (h(a_1), \ldots, h(a_n)) \in \text{NCT}(A_2).
\]

It’s tempting to say that the NC congruences of \( A \) should be those contained in the center of \( A \) but we aren’t sure whether that is always the case yet.
The first functor: Open serenation

- Our first construction gives a functor

\[ \text{OSer}_n : \text{NCAQ}_n \to \text{SMfld}_n. \]

- We define

\[ \text{Sim}_n : \text{NCAQ}_n \to \text{PMfld}_n \]

similarly to our previous example for \( n = 2 \).

- We define \( \text{In}(\mathbf{A}) \) to consist of all entries in noncommuting tuples of \( \mathbf{A} \) and \( \text{Out}(\mathbf{A}) \) to consist of all \( f(a_1, \ldots, a_n) \) where \( (a_1, \ldots, a_n) \in \text{NCT}(\mathbf{A}) \).
The first functor: Open serenade

- We set

\[ \text{Sim}(A) := \{ a \mid a \in \text{In}(A) \} \cup \{ \overline{a} \mid a \in \text{Out}(A) \} \]

and

\[ \text{SimFace}(A) := \bigcup_{a \in \text{NCT}(A)} \text{Sb} \left( \{ a_1, \ldots, a_n, f(a) \} \right) \]

- We define

\[ \text{Sim}_n(A) := (\text{Sim}(A), \text{SimFace}(A)). \]
The first functor: Open serenade

We create \( \text{OSer}_n(A) \) by taking the open geometric realization of \( \text{Sim}_n(A) \) (basically all but the \((n - 2)\)-skeleton of the open geometric realization) and then equipping it with a smooth atlas.

The standard open bipyramid (or just bipyramid) in \( \mathbb{R}^n \) is

\[
\text{Bipy}_n := \text{OCvx} \left( \left\{ (0, \ldots, 0), \left( \frac{2}{n}, \ldots, \frac{2}{n} \right) \right\} \cup \{ e_1, \ldots, e_n \} \right)
\]

where \( e_i \) is the \(i^{th}\) standard basis vector of \( \mathbb{R}^n \).
The first functor: Open serenade

- Given an alternating $n$-quasigroup $A$ and $a = (a_1, \ldots, a_n) \in NCT(A)$ the serene chart of input type for $a$ is

$$\phi_a : \text{Bipy}_n \rightarrow \text{OSer}_n(A).$$

- We set

$$\phi_a(u_1, \ldots, u_n) := \sum_{i=1}^{n} u_i a_i + \left(1 - \sum_{i=1}^{n} u_i\right) \overline{f(a)}$$

when $\sum_{i=1}^{n} u_i \leq 1$.

- Otherwise,

$$\phi_a(u_1, \ldots, u_n) := \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{n-2}{2} u_i - \sum_{j \neq i} u_j\right) a_i +$$

$$\left(-1 + \sum_{i=1}^{n} u_i\right) \overline{f(a')}.$$
The first functor: Open serenation

There are also serene charts of output type, where are defined similarly.

We set

$$(\text{OSer}_n(A), \tau) := (\text{OGeo}_n \circ \text{Sim}_n)(A).$$

We then define

$$\text{OSer}_n(A) := (\text{OSer}_n(A), \tau, \text{SerAt}_n(A))$$

where

$$\text{SerAt}_n(A) := \bigcup_{a \in \text{NCT}(A)} \{\phi_a, \bar{\phi}_a\}.$$
The first functor: Open serenade

- The incidence graph of the facets of $\text{Sim}(A)$ for the ternary quasigroup $A$ from the previous example is pictured.
The first functor: Open serenade

The 1-skeleton of Sim(A) for the ternary quasigroup A from the previous example is pictured.
The first functor: Open serenade

- One may verify that $\text{OSer}(A)$ is a 3-sphere minus the graph pictured previously, which is homotopy equivalent to the wedge sum of 21 circles.
The second functor: Serenation

- For any alternating quasigroup $A$ we may equip $\text{OSer}(A)$ with a Riemannian metric in a functorial manner which makes $\text{OSer}(A)$ flat.
- We then define a \textit{Euclidean metric completion functor} $\text{EuCmplt}: \text{Riem}_n \rightarrow \text{Mfld}_n$

  which assigns to a Riemannian manifold $(M, g)$ the topological manifold consisting of all points in the metric completion of $M$ which are locally Euclidean.
The second functor: Serenation

The serenation functor

$$\text{Ser}_n: \text{NCAQ}_n \to \text{Mfld}_n$$

is given by

$$\text{Ser}(A) := \text{EuCmplt}(\text{OSer}(A), g)$$

where $g$ is the standard metric on $\text{OSer}(A)$.

In the previous example of the ternary quasigroup $A$ we find that $\text{Ser}_3(A)$ is the 3-sphere.
The second functor: Serenation

**Definition (Serene manifold)**

We say that a connected orientable $n$-manifold $\mathbf{M}$ is *serene* when there exists some alternating $n$-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is a component of $\text{Ser}(\mathbf{A})$. 
The second functor: Serenation

Theorem (A., Yoo (2021))

*Every connected orientable triangulable $n$-manifold is serene.*
The second functor: Serenation

Theorem (A., Yoo (2021))

*Every connected orientable triangulable n-manifold is serene.*

- Consider a triangulation of the given manifold $\mathbf{M}$.
- Subdivide each facet in a manner I will draw off to the side.
- We have that $\mathbf{M}$ is homeomorphic to a corresponding component of the serenation of a quotient of the free alternating $n$-quasigroup whose generators are the vertices of the subdivided triangulation.
The Evans Conjecture and Latin cubes

Definition (Quasifinite manifold)

We say that a connected compact orientable smooth $n$-manifold $M$ is *quasifinite* when there exists a finite alternating $n$-quasigroup $A$ such that $M$ is homeomorphic to a component of $\text{Ser}(A)$.

- Is every connected compact orientable smooth manifold quasifinite?
The Evans Conjecture and Latin cubes

Definition (Partial Latin cube)

Given a set $A$ and some $n \in \mathbb{N}$ we say that $\theta \subset A^{n+1}$ is a partial Latin $n$-cube when for each $i \in [n]$ and each

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in A^n$$

there exists at most one $a_i \in A$ so that

$$(a_1, \ldots, a_{n+1}) \in \theta.$$
The Evans Conjecture and Latin cubes

Evans conjectured that each partial Latin square (i.e. a partial Latin cube $\theta \subset A^{2+1}$) with $|A| = k$ and $|\theta| \leq n - 1$ could be filled in so as to obtain a complete Latin square $\psi \subset A^3$ with $\theta \subset \psi$ and $|\psi| = k^2$.

This was proven to be true by Smetaniuk in 1981.

Similar results are known for special classes of higher-dimensional Latin cubes.
The Evans Conjecture and Latin cubes

- In general a *complete Latin n-cube* is the graph of an $n$-quasigroup operation.

- We say that a partial Latin $n$-cube is *alternating* when we have for each $\alpha \in \text{Alt}_n$ that if

$$ (a_1, \ldots, a_n, b_1) \in \theta $$

and

$$ (a_{\alpha(1)}, \ldots, a_{\alpha(n)}, b_2) \in \theta $$

then $b_1 = b_2$.

- Given a finite partial alternating Latin cube $\theta \subset A^{n+1}$ does there always exist a finite complete alternating Latin cube $\psi \subset B^{n+1}$ such that $\theta \subset \psi$?
The Evans Conjecture and Latin cubes

- We don’t ask for any particular relationship between $|\theta|$ and $|B|$, so this is in one sense a weaker question than the Evans Conjecture. That is, we may add many new elements to $A$ in order to complete our Latin cube, as long as we only add finitely many.

- We have a corollary of the Evans Conjecture for the $n = 2$ case.

**Corollary**

*Every connected compact orientable surface is a component of the serenation of some finite binary quasigroup.*
Mark Herman and Jonathan Pakianathan. “On a canonical construction of tessellated surfaces from finite groups”. In: Topology Appl. 228 (2017), pp. 158–207. ISSN: 0166-8641