Multiplayer rock-paper-scissors

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Introduction

In the summer of 2017 I lived in a cave in Yosemite National Park.

While there I wanted to explain to my friends that I study universal algebra.

I realized that rock-paper-scissors was a particularly simple way to do that.
We will view the game of RPS as a magma $A := (A, f)$. We let $A := \{r, p, s\}$ and define a binary operation $f: A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

\[
\begin{array}{ccc}
  r & p & s \\
  r & r & p & r \\
  p & p & p & s \\
  s & r & s & s \\
\end{array}
\]
I also realized that I wanted to be able to play with many of my friends at the same time.

Naturally, this led me to study the varieties generated by hypertournament algebras.
Talk outline

- Definition of RPS and PRPS magmas
- A numerical constraint relating arity and order
- Regular RPS magmas
- Hypertournaments
- A generation result
- Automorphisms and congruences of regular RPS magmas
- The search for a basis of the variety generated by tournament algebras
The game RPS is

1. conservative,

2. essentially polyadic,

3. strongly fair, and

4. nondegenerate.

These are the properties we want for a multiplayer game, as well.
What does a multiplayer game mean?

- Suppose we have an *n-ary magma* \( A := (A, f) \) where \( f: A^n \to A \).
- The *selection game* for \( A \) has \( n \) players, \( p_1, p_2, \ldots, p_n \).
- Each player \( p_i \) simultaneously chooses an item \( a_i \in A \).
- The winners of the game are all players who chose \( f(a_1, \ldots, a_n) \).
Properties of RPS: conservativity

- We say that an operation \( f : A^n \rightarrow A \) is **conservative** when for any \( a_1, \ldots, a_n \in A \) we have that \( f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\} \).
- We say that \( A \) is conservative when each round has at least one winning player.
We say that an operation \( f: A^n \to A \) is *essentially polyadic* when there exists some \( g: Sb(A) \to A \) such that for any \( a_1, \ldots, a_n \in A \) we have \( f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\}) \).

We say that \( A \) is essentially polyadic when a round’s winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item (as long as it was chosen at least once).
Properties of RPS: strong fairness

- Let $A_k$ denote the members of $A^n$ which have $k$ distinct components for some $k \in \mathbb{N}$.
- We say that $f$ is strongly fair when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $|f^{-1}(a) \cap A_k| = |f^{-1}(b) \cap A_k|$.
- We say that $A$ is strongly fair when each item has the same chance of being the winning item when exactly $k$ distinct items are chosen for any $k \in \mathbb{N}$. 
Properties of RPS: nondegeneracy

- We say that $f$ is nondegenerate when $|A| > n$.
- In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components.
- If $A$ is essentially polyadic with $|A| \leq n$ it is impossible for $A$ to be strongly fair unless $|A| = 1$. 
The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic.

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The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

\[
\begin{array}{c|ccccc}
& r & p & s & v & l \\
\hline
r & r & p & r & v & r \\
p & p & p & s & p & l \\
s & r & s & s & v & s \\
v & v & p & v & v & l \\
l & r & l & s & l & l \\
\end{array}
\]
Result for two-player games

The only “valid” RPS variants for two players use an odd number of items.

**Proposition**

Let $A$ be a selection game with $n = 2$ which is essentially polyadic, strongly fair, and nondegenerate and let $m := |A|$. We have that $m \neq 1$ is odd. Conversely, for each odd $m \neq 1$ there exists such a selection game.

**Proof.**

We need $m \mid \binom{m}{2}$.
PRPS magmas

Definition (PRPS magma)

Let $A := (A, f)$ be an $n$-ary magma. When $A$ is essentially polyadic, strongly fair, and nondegenerate we say that $A$ is a PRPS magma (read “pseudo-RPS magma”). When $A$ is an $n$-magma of order $m \in \mathbb{N}$ with these properties we say that $A$ is a PRPS($m, n$) magma. We also use PRPS and PRPS($m, n$) to indicate the classes of such magmas.
Result for multiplayer games

Theorem

Let $A \in \text{PRPS}(m, n)$ and let $\varpi(m)$ denote the least prime dividing $m$. We have that $n < \varpi(m)$. Conversely, for each pair $(m, n)$ with $m \neq 1$ such that $n < \varpi(m)$ there exists such a magma.

Proof.

We need $m \mid \gcd \left( \{ \binom{m}{2}, \ldots, \binom{m}{n} \} \right)$. \qed
RPS magmas

**Definition (RPS magma)**

Let $A := (A, f)$ be an $n$-ary magma. When $A$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $A$ is an RPS *magma*. When $A$ is an $n$-magma of order $m$ with these properties we say that $A$ is an RPS$(m, n)$ *magma*. We also use RPS and RPS$(m, n)$ to indicate the classes of such magmas.

Both the original game of rock-paper-scissors and the game rock-paper-scissors-Spock-lizard are RPS magmas. The French variant of rock-paper-scissors is not even a PRPS magma.
A game for three players

- We now show how to construct a game for three players.
- This will be a ternary RPS magma \((A, f: A^3 \rightarrow A)\).
- Since \(n = 3\) in this case and we require that \(n < \omega(m)\) we must have that \(|A| \geq 5\).
- Our construction will use the left-addition action of \(\mathbb{Z}_5\) on itself.
- We will produce an operation \(f: \mathbb{Z}_5^3 \rightarrow \mathbb{Z}_5\) which is essentially polyadic with \(w + f(x, y, z) = f(w + x, w + y, w + z)\) for any \(w \in \mathbb{Z}_5\).
- Thus, we need only define \(f\) on a representative of each orbit of \((\mathbb{Z}_5)_1\), \((\mathbb{Z}_5)_2\), and \((\mathbb{Z}_5)_3\) under this action of \(\mathbb{Z}_5\).
A game for three players

First we list the orbits of \((\mathbb{Z}_5)_1\), \((\mathbb{Z}_5)_2\), and \((\mathbb{Z}_5)_3\) under this action of \(\mathbb{Z}_5\).

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A game for three players

Next, we choose a representative for each orbit, say the first one in each row of this table.

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A game for three players

Choose from each representative a particular element. For example, if our representative is 013 we may choose 0 as our special element. We also could have chosen 1 or 3, but not 2 or 4.

<table>
<thead>
<tr>
<th>0 ↔ 0</th>
<th>01 ↔ 1</th>
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A game for three players

Use the left-addition action of $\mathbb{Z}_5$ to extend these choices to all members of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$.

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<thead>
<tr>
<th>0 $\leftrightarrow$ 0</th>
<th>01 $\leftrightarrow$ 1</th>
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A game for three players

We can read off a definition for the operation \( f: \mathbb{Z}_5^3 \to \mathbb{Z}_5 \) from this table. For example, we take \( 24 \mapsto 2 \) to indicate that

\[
f(2, 4, 4) = f(4, 2, 4) = f(4, 4, 2) = f(4, 2, 2) = f(2, 4, 2) = f(2, 2, 4) = 2.
\]

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A game for three players

The Cayley table for the 3-magma \( \mathbf{A} := (\mathbb{Z}_5, f) \) obtained from this choice of \( f \) is given below.

\[
\begin{array}{cccc|cccc|cccc}
0 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 4 \\
1 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & 1 & 2 & 1 & 4 & 1 & 0 & 2 & 2 & 1 & 1 \\
2 & 0 & 0 & 2 & 4 & 2 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 2 & 2 & 3 & 2 \\
3 & 3 & 0 & 2 & 3 & 3 & 3 & 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 & 3 & 3 & 2 \\
4 & 0 & 4 & 4 & 3 & 0 & 4 & 4 & 4 & 1 & 3 & 4 & 4 & 1 & 2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc|cccc|cccc}
3 & 0 & 1 & 2 & 3 & 4 & 4 & 0 & 1 & 2 & 3 & 4 \\
0 & 3 & 0 & 2 & 3 & 3 & 0 & 0 & 4 & 4 & 3 & 0 \\
1 & 0 & 1 & 1 & 1 & 3 & 1 & 4 & 4 & 1 & 3 & 4 \\
2 & 2 & 1 & 3 & 3 & 2 & 2 & 4 & 1 & 2 & 2 & 2 \\
3 & 3 & 1 & 3 & 3 & 4 & 3 & 3 & 3 & 2 & 4 & 4 \\
4 & 3 & 3 & 2 & 4 & 4 & 4 & 0 & 4 & 2 & 4 & 4 \\
\end{array}
\]
**Definition (α-action magma)**

Fix a group $G$, a set $A$, and some $n < |A|$. Given a regular group action $\alpha: G \to \text{Perm}(A)$ such that each of the $k$-extensions of $\alpha$ is free for $1 \leq k \leq n$ let $\Psi_k := \left\{ \text{Orb}(U) \mid U \in \binom{A}{k} \right\}$ where $\text{Orb}(U)$ is the orbit of $U$ under $\alpha_k$. Let $\beta := \{\beta_k\}_{1 \leq k \leq n}$ be a sequence of choice functions $\beta_k: \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma := \{\gamma_k\}_{1 \leq k \leq n}$ be a sequence of functions $\gamma_k: \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Let $g: \text{Sb}_{\leq n}(A) \to A$ be given by $g(U) := (\alpha(s))(\gamma_k(\psi))$ when $U = (\alpha_k(s))(\beta_k(\psi))$. Define $f: A^n \to A$ by $f(a_1, \ldots, a_n) := g(\{a_1, \ldots, a_n\})$. The $\alpha$-action magma induced by $(\beta, \gamma)$ is $A := (A, f)$. 

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**α-action magmas**
\( \alpha \)-action magmas are RPS magmas

<table>
<thead>
<tr>
<th>Theorem</th>
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<tr>
<td>Let ( A ) be an ( \alpha )-action magma induced by ((\beta, \gamma)). We have that ( A \in \text{RPS} ).</td>
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<table>
<thead>
<tr>
<th>Definition (Regular RPS magma)</th>
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<tr>
<td>Let ( G ) be a nontrivial finite group and fix ( n &lt; \omega(</td>
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**Hypergraphs**

**Definition (Pointed hypergraph)**

A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph $(S, \sigma)$ and a map $g: \sigma \rightarrow S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map $g$ is called a *pointing* of $(S, \sigma)$.

**Definition (n-complete hypergraph)**

Given a set $S$ we denote by $S_n$ the *n-complete hypergraph* whose vertex set is $S$ and whose edge set is $\bigcup_{k=1}^{n} \binom{S}{k}$. 
**Definition (Hypertournament)**

An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (\mathbb{T}, \tau, g)$ where $(\mathbb{T}, \tau) = S_n$ for some set $S$.

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<tr>
<th>$U$</th>
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<td>$g(U)$</td>
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<td>$g(U)$</td>
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**RPS(5, 3) example**
Hypertournament magmas

Definition (Hypertournament magma)

Given an \( n \)-hypertournament \( T := (T, \tau, g) \) the hypertournament magma obtained from \( T \) is the \( n \)-magma \( A := (T, f) \) where for \( u_1, \ldots, u_n \in T \) we define

\[
f(u_1, \ldots, u_n) := g(\{u_1, \ldots, u_n\}).
\]

Definition (Hypertournament magma)

A hypertournament magma is an \( n \)-magma which is conservative and essentially polyadic.
Tournaments are the $n = 2$ case of a hypertournament.

Hedrlín and Chvátal introduced the $n = 2$ case of a hypertournament magma in 1965.

There has been a lot of work on varieties generated by tournament magmas. See for example the survey by Crvenković et al. (1999).
Proposition

Let $n > 1$. We have that $\text{RPS}_n \subsetneq \text{PRPS}_n$, $\text{RPS}_n \subsetneq \text{Tour}_n$, and neither of $\text{PRPS}_n$ and $\text{Tour}_n$ contains the other. Moreover, $\text{RPS}_n = \text{PRPS}_n \cap \text{Tour}_n$. 
A generation result

- We denote by $\mathcal{T}_n$ the variety of algebras generated by $\text{Tour}_n$.
- This is equivalent to having

$$\mathcal{T}_n = \text{HSP}(\text{Tour}_n) = \text{Mod}($$Id($\text{Tour}_n$)).$$

- Similarly, we define $\mathcal{R}_n$ to be the variety of algebras generated by $\text{RPS}_n$. 
A generation result

**Theorem**

Let $n > 1$. We have that $T_n = R_n$. Moreover $T_n$ is generated by the class of finite regular $\text{RPS}_n$ magmas.

**Proof.**

Every finite hypertournament can be embedded in a finite regular balanced hypertournament.
A generation result

- Trivially we have that $\mathcal{R}_n \leq \mathcal{T}_n$.
- Since $n$-hypertournament magmas are conservative we have that $\text{Tour}_n \models \epsilon$ if and only if each $n$-hypertournament magma of order $m$ models epsilon, where $m$ is the number of variables appearing in $\epsilon$.
- It then suffices to show that each finite $n$-hypertournament magma belongs to $\mathcal{R}_n$.
- It would be very convenient if each finite $n$-hypertournament embedded into the hypertournament associated to a finite regular RPS magma.
- This turns out to be the case.
A generation result

- Note that in a regular binary RPS magma $G_2(\beta, \gamma)$ we have that
  \[ f(e, x) = xf(x^{-1}, e) \]
  so exactly one of $f(e, x) = e$ or $f(x^{-1}, e) = e$ holds.

- Note also that the orbit of $\{x, y\}$ contains $\{e, x^{-1}y\}$ and $y^{-1}x$, $e$, where $x^{-1}y$ and $y^{-1}x$ are inverses.

- We need then only define a map $\lambda$ specifying for each pair of inverses $\{x, x^{-1}\}$ whether $f(e, x) = e$ or $f(e, x^{-1}) = e$ in order to specify $G_2(\beta, \gamma)$.

- We can think of $\lambda(\{x, x^{-1}\})$ as choosing the «positive direction» with respect to $x$ and $x^{-1}$. 
A generation result

In order to do this in general we need an \( n \)-ary analogue of inverses.

**Definition (Obverse \( k \)-set)**

Given \( n > 1 \), a nontrivial finite group \( G \) with \( n < \omega(|G|) \), \( 1 \leq k \leq n-1 \), and \( U, V \in \binom{G \setminus \{e\}}{k} \) we say that \( V \) is an *obverse* of \( U \) when \( U = \{a_1, \ldots, a_k\} \) and there exists some \( a_i \in U \) such that \( V = \{a_i^{-1}\} \cup \{a_i^{-1}a_j \mid i \neq j \} \). We denote by \( \text{Obv}(U) \) the set consisting of all obverses \( V \) of \( U \), as well as \( U \) itself.

The obverses of a set \( U \) are the nonidentity elements in the members of \( \text{Orb}(U \cup \{e\}) \setminus (U \cup \{e\}) \) which contain \( e \).
A generation result

In order to specify $G_n(\beta, \gamma)$ it suffices to choose the member
$\{a_1, \ldots, a_k\}$ of each collection of obverses for which
$f(e, \ldots, e, a_1, \ldots, a_k) = e.$

**Definition (n-sign function)**

Given $n > 1$ and a nontrivial group $G$ with $n < \omega(|G|)$ let $Sgn_n(G)$ denote the set of all choice functions on

$$\left\{ \text{Obv}(U) \mid (\exists k \in \{1, \ldots, n - 1\}) \left(U \in \binom{G \setminus \{e\}}{k}\right) \right\}.$$

We refer to a member $\lambda \in Sgn_n(G)$ as an *n-sign function* on $G$.

We then write $G_n(\lambda)$ instead of $G_n(\beta, \gamma)$.
Now we can give the embedding which finishes our proof that $T_n = R_n$.

Consider a finite hypertournament $T := (T, \tau, g)$.

Take $G := \bigoplus_{u \in T} \mathbb{Z}_{\alpha_u}$ where $n < \omega(\alpha_u)$ and $\mathbb{Z}_{\alpha_u} = \langle u \rangle$.

We define an $n$-sign function $\lambda \in \text{Sgn}_n(G)$.

When $g(\{u_1, \ldots, u_k\}) = u_1$ we define

$$\lambda(\text{Obv}(\{ u_i - u_1 \mid i \neq 1 \})) := \{ u_i - u_1 \mid i \neq 1 \}.$$ 

Any values may be chosen for other orbits.

The $n$-hypertournament corresponding to $G_n(\lambda)$ contains a copy of $T$. 

A generation result
A generation result

- We have now seen that the finite regular RPS $n$-magmas generate $\mathcal{T}_n = \mathbf{V}(\text{Tour}_n)$.

- In particular we need only use magmas of the form $G_n(\lambda)$ where:
  1. $G = \mathbb{Z}_{\kappa(n)}^m$ where $\kappa(n)$ is the least prime strictly greater than $n$ or
  2. $G = \mathbb{Z}_{\alpha(m)}^m$ where $\alpha(m) := \prod_{k=\ell}^{m+\ell-1} p_k$ where $p_k$ is the $k^{th}$ prime and $\kappa(n) = p_\ell$.

- In particular, we have that $\mathcal{T}_2$ is generated by regular RPS magmas of the form $(\mathbb{Z}_3^m)_2(\lambda)$. 
Automorphisms

Proposition

Let $A := G_n(\lambda)$ be a regular RPS magma. There is a canonical embedding of $G$ into $\text{Aut}(A)$.

Proof.

By construction.
Exceptional automorphisms

Proposition

For each arity $n \in \mathbb{N}$ with $n \neq 1$ and each group $G$ of composite order $m \in \mathbb{N}$ with $n < \varpi(m)$ there exists a regular RPS$(m, n)$ magma $A := G_n(\lambda)$ such that $|\text{Aut}(A)| > |G|$.

Proof.

Count the members of RPS$(G, n)$ (there are $\prod_{k=1}^{n} k_{m}^{(m)}$) and arrive at a contradiction were there no exceptional automorphisms.
Exceptional automorphisms

Proposition

For each arity $n \in \mathbb{N}$ and each odd prime $p$ such that $1 \neq n \leq p - 2$ there exists a regular RPS($p, n$) magma $A := (\mathbb{Z}_p)_n(\lambda)$ such that $|\text{Aut}(A)| > |G|$.

Proof.

Multiplication by a primitive root modulo $p$ yields an automorphism for an appropriate choice of $\lambda$. \qed
 Proposition

For each odd prime $p$ and any $\lambda \in \text{Sgn}_{p-1}(\mathbb{Z}_p)$ we have that \( \text{Aut}((\mathbb{Z}_p)_{p-1}(\lambda)) \cong \mathbb{Z}_p. \)

 Corollary

Given an odd prime $p$ the number of isomorphism classes of magmas of the form \((\mathbb{Z}_p)_{p-1}(\lambda)\) is

\[
\prod_{k=1}^{p-1} k^{p \frac{1}{p} \binom{k}{p} - 1}.
\]

For $p = 3$ we have 1, for $p = 5$ we have 6, and for $p = 7$ we have 2073600.
Congruences

Theorem

Let $\theta \in \text{Con}(A)$ for a regular RPS$(m, n)$ magma $A := G_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $H \leq G$.

One can show by using 2-divisibility that the principal congruence $\theta := \text{Cg}((e, a))$ has only one nontrivial class, which is $e/\theta$. This class contains $\text{Sg}^G(\{a\})$. 
Theorem

Let \( \theta \in \text{Con}(A) \) for a regular RPS\((m, n)\) magma \( A := G_n(\lambda) \). Given any \( a \in A \) we have that \( a/\theta = aH \) for some subgroup \( H \leq G \).

- Any congruence \( \theta \in \text{Con}(A) \) has for \( e/\theta \) a union of cyclic subgroups of \( G \). Suppose that \( a, b \in e/\theta \) and \( ab \notin e/\theta \).
- Note that \( \theta \geq \text{Cg}(\{(e, a), (e, b^{-1})\}) \). Observe that

\[
\text{Cg}(\{(e, a), (e, b^{-1})\}) = b^{-1} \text{Cg}(\{(b, ba), (b, e)\}) \\
\geq b^{-1} \text{Cg}(\{(e, ba)\}) \\
\geq b^{-1} \text{Cg}(\{(e, baba)\}) \\
\geq \text{Cg}(\{(b^{-1}, aba)\})
\]

so we have that \( e/\theta \) contains \( aba \).
Theorem

Let $\theta \in \text{Con}(A)$ for a regular RPS$(m, n)$ magma $A := G_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $H \leq G$.

- We have $\langle a \rangle, \langle b \rangle \subset e/\theta$ and $ab \not\in e/\theta$ yet $aba \in e/\theta$.
- Since $\theta$ is a congruence either $ab$ dominates everything in $e/\theta$ ($f(ab, x) = ab$ for all $x \in e/\theta$, which we write as $ab \rightarrow x$) or everything in $e/\theta$ dominates $ab$.
- In the former case, we have $ab \rightarrow aba$ so $e \rightarrow a$.
- We also have $ab \rightarrow e$ so $e \rightarrow b^{-1}a^{-1}$.
- This implies that $b^{-1} \rightarrow b^{-1}a^{-1}$ and hence $e \rightarrow a^{-1}$, which is impossible since $e \rightarrow a$.
- The argument in the latter case is identical.
- Thus, $e/\theta$ is a subgroup of $G$. 

Definition (λ-convex subgroup)

Given a group $G$, an $n$-sign function $\lambda \in Sgn_n(G)$, and a subgroup $H \leq G$ we say that $H$ is $\lambda$-convex when there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \text{Con}(G_n(\lambda))$. 
Proposition

Let $G$ be a finite group of order $m$ and let $n < \varpi(m)$. Take $\lambda \in \text{Sgn}_n(G)$ and $H \leq G$. The following are equivalent:

1. The subgroup $H$ is $\lambda$-convex.
2. There exists a congruence $\psi \in \text{Con}(G_n(\lambda))$ such that $e/\psi = H$.
3. Given $1 \leq k \leq n - 1$ and $b_1, \ldots, b_k \notin H$ either $e \to \{b_1 h_1, \ldots, b_k h_k\}$ for every choice of $h_1, \ldots, h_k \in H$ or $\{b_1 h_1, \ldots, b_k h_k\} \to e$ for every choice of $h_1, \ldots, h_k \in H$. 
\textbf{Theorem}

Suppose that $H, K \leq G$ are both $\lambda$-convex. We have that $H \leq K$ or $K \leq H$. 
Definition (λ-coset poset)

Given $\lambda \in \text{Sgn}_n(G)$ set

$$P_\lambda := \{ aH \mid a \in G \text{ and } H \text{ is } \lambda\text{-convex} \}$$

and define the $\lambda$-coset poset to be $P_\lambda := (P_\lambda, \subseteq)$. 
Dilworth showed that the maximal antichains of a finite poset form a distributive lattice.

Freese (1974) gives a succinct treatment of this.

Given a finite poset $\mathbf{P} := (P, \leq)$ let $L(P)$ be the lattice whose elements are maximal antichains in $\mathbf{P}$ where if $U, V \in L(P)$ then we say that $U \leq V$ in $L(P)$ when for every $u \in U$ there exists some $v \in V$ such that $u \leq v$ in $\mathbf{P}$.

**Theorem**

We have that $\text{Con}(G_n(\lambda)) \cong L(P_{\lambda})$. 
The search for a basis

By the year 2000 Ježek, Marković, Maróti, and McKenzie had shown that $\mathcal{T}_2$ was not finitely based.

To this author’s knowledge no equational base for $\mathcal{T}_2$ has ever been described (aside from trivialities like taking $\text{Id}(\text{Tour}_2)$).

Recall that an identity $\epsilon$ in $m$ variables holds in $\mathcal{T}_2$ if and only if it holds in each tournament magma of order $m$.

We can use our generation result to see that $\mathcal{T}_2 \models \epsilon$ if and only if $\epsilon$ holds in each regular $\text{RPS}_2$ magma of the form $(\mathbb{Z}_3^m)_2(\lambda)$.

These magmas are much larger than tournaments of order $m$, but we may have a better chance of understanding their structure and hence their equational theories.
Thank you.