Relational Structures as Directed Hypergraphs

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Abstract

In mathematics it is often useful to encode abstract algebraic structure as geometric structure, both to aid one's intuition and to permit the application of geometric methods to ostensibly algebraic questions. We examine one such encoding scheme, which we use to obtain directed graphs from unary operations. These graphs are then used to obtain lower bounds on the number of solutions to certain equations. We then apply a generalization of this technique to obtain 2-dimensional simplicial complexes from binary operations. These simplicial complexes permit the application of topological methods to the study of group structure.

Theorem (Solution counting via operation digraphs)

Let V be an ordered finite set of elements and let $\{f_q : V \to V\}_{q \in I}$ be an indexed collection of functions. Let $G_q = G(V, E_q)$ denote the operation digraph for f_q, and let A_q be the adjacency matrix for G_q under the given ordering for V. If $S = \{s_n\}_{n=1}^{k}$ is a finite sequence of k elements of I and $y = v_j$ is a fixed element of V we have that the number of $x \in V$ for which $f^S(x) = y$ is exactly

$$\sum_{i=1}^{|V|} \left(\prod_{n=1}^{k} (A_{s_n}) \right)_{ij}.$$

Corollary (A lower bound on the number of solutions)

The number of solutions (x, y) to an equation of the form $f^S(x) = y$ is bounded below by the rank of the matrix $\prod_{n=1}^{k}(A_{s_n})$. By Sylvester's Rank Inequality and the above counting theorem we have that

$$\operatorname{rank}\left(\prod_{i \in I} A_i \right) \geq \left(\sum_{i \in I} \operatorname{rank}(A_i) \right) - (|I| - 1) \dim V.$$

Definition (Operation digraph)

Let V be an ordered set of elements (or vertices) endowed with a unary operation $f : V \to V$ and let $x, y \in V$. The *operation digraph of f*, written G_f, is given by $G_f = G(V, E)$ where $E = \{(x, y) | f(x) = y\}$.

Definition (Adjacency matrix)

Let $G(V, E)$ be a digraph, let $|V| = n$, and fix an order on the vertex set V. The *adjacency matrix* A for G under the given order on V is the $n \times n$ matrix whose ij-entry is 1 if there is an edge in G from v_i to v_j and 0 otherwise.

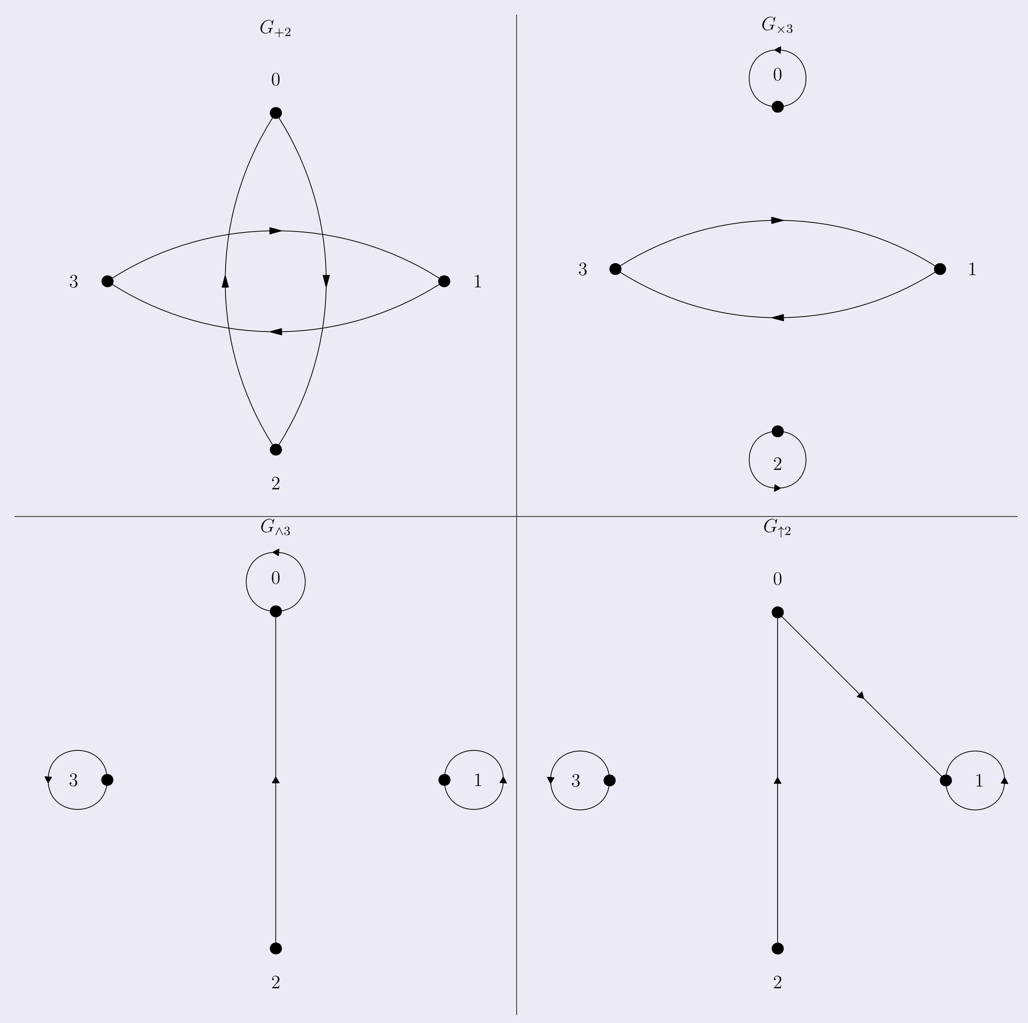
Application (Bounding the number of solutions to an equation)

We apply our corollary to obtain a lower bound on the number of solutions to

$$\left((3(x+2))^3\right)^{\left((3(x+2))^3\right)} = y$$

over $V = \mathbb{Z}_4$. Let $f_1(x) = x + 2$, $f_2(x) = 3x$, $f_3(x) = x^3$, and $f_4(x) = x^x$. The equation under consideration can be rewritten as $f^S(x) = y$, where S is the sequence $(1, 2, 3, 4)$. We see that the corollary does indeed apply here.

Operation digraphs



Adjacency matrices

We obtain the following adjacency matrices for the above operation digraphs.

$$M_1 = A_{+2} = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{bmatrix} \quad M_2 = A_{\times 3} = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix}$$

$$M_3 = A^R_{\wedge 3} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix} \quad M_4 = A^R_{\uparrow 2} = \begin{bmatrix} 0&1&0&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}$$

Minimum rank computation

Since $\operatorname{rank}(A_{+2}) = \operatorname{rank}(A_{\times 3}) = 4$ and $\operatorname{rank}\left(A^R_{\wedge 3}\right) = \operatorname{rank}\left(A^R_{\uparrow 2}\right) = 3$, we have by way of our corollary that

$$\begin{aligned} \operatorname{rank}\left(\prod_{n=1}^{4} M_n \right) &\geq \left(\sum_{n=1}^{4} \operatorname{rank}(M_n) \right) - (|I| - 1)|V| \\ &= (4 + 4 + 3 + 3) - (4 - 1)4 \\ &= 2. \end{aligned}$$

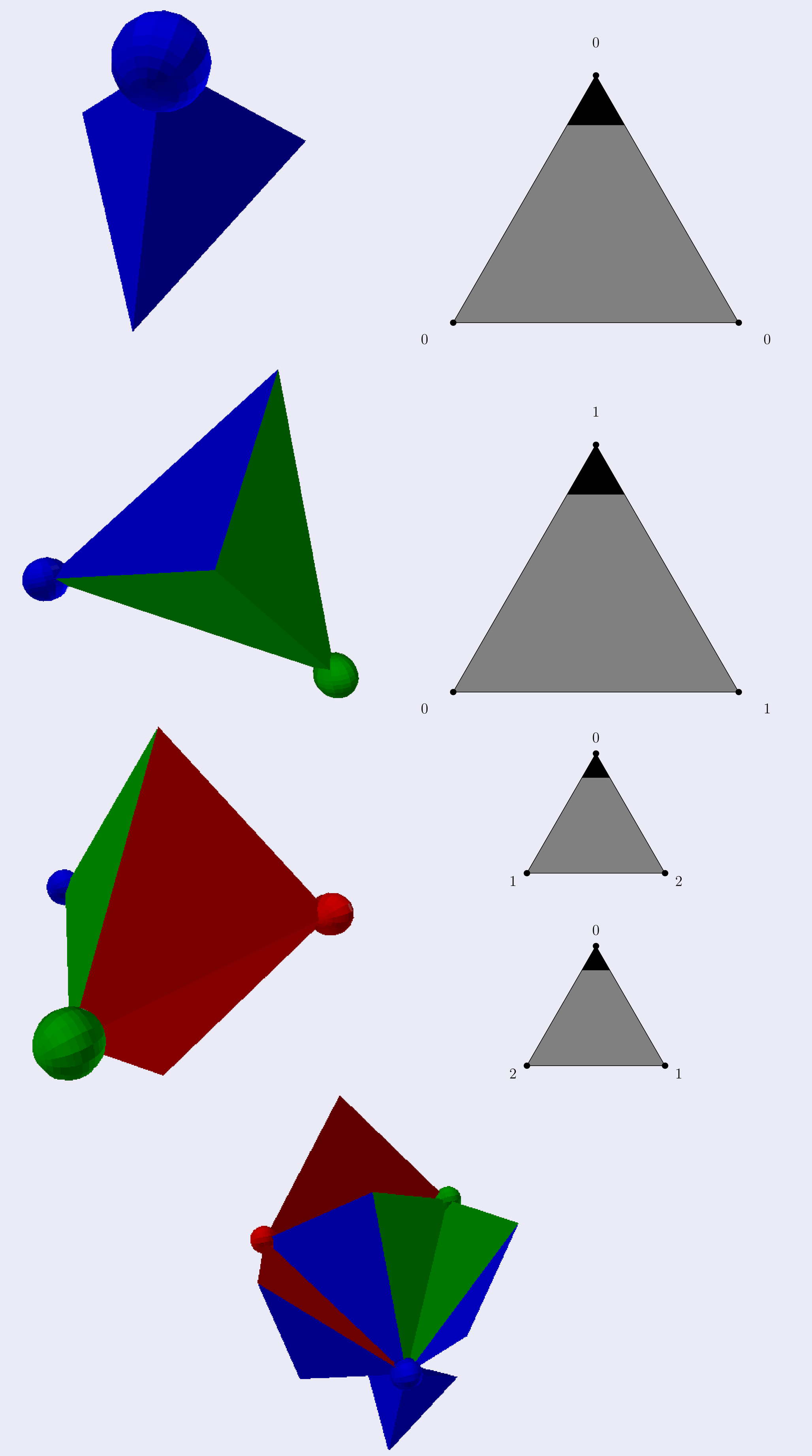
Thus, there are at least two pairs $(x, y) \in V^2$ such that $f^S(x) = y$.

Definition (Relation hypergraph)

Let $f \subset S^n$ be an n-ary relation on S. Then define the *relation hypergraph* $\Gamma(f) = (S, f)$. Given a relational structure $\mathbf{A} = (S, F = \{f_i\}_{i \in I})$ from $\mathbf{Rel}_\rho(S)$ define

$$\Gamma\mathbf{A} = (\Gamma(f_i))_{i \in I}.$$

The group of order 3



Acknowledgements

I would like to thank professors Jonathan Pakianathan and Mark Herman of the University of Rochester Department of Mathematics for advising me on this project, the Ronald E. McNair Post-Baccalaureate Achievement Program and the Kearns Center at the University of Rochester for funding and supporting this research, and the University of Rochester Office of Undergraduate Research for funding my travel. I would also like to thank the Nebraska Conference for Undergraduate Women in Mathematics for hosting this event.