Exercises for the Rochester AMS Grad Student Chapter Problem Solving Seminar

Nikolaos Chatzikonstantinou

28th October, 2019

(Note: The exercises below are independent of each other)

Let \( n \in \mathbb{N}, n \geq 2 \), and \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) with the surface measure \( \sigma \) defining integration on the unit sphere. A set \( \Lambda \subset S^{n-1} \) is \( \delta \) - separated for some \( \delta > 0 \) if \( x, y \in \Lambda \) and \( x \neq y \) implies \( |x - y| \geq \delta \). The characteristic function is denoted by \( 1_Q(x) = 1 \) if \( x \in Q \) and \( 0 \) if \( x \notin Q \).

The Stein-Thomas restriction conjecture [S] is

\[
\| \hat{f} \sigma \|_{L^p} \lesssim \| f \|_{L^2(\sigma)}, \quad p \geq \frac{2(n+1)}{n-1}.
\]  

(\(^*\))

An equivalent discrete version (see [BD]) is that for all \( 0 < \delta \leq 1 \), for all \( \Lambda \subset S^{n-1} \) sets which are \( \delta^{1/2} \) separated, all \( a_\xi \in \mathbb{C} \) and all balls \( B_R \) (of any center) where \( R \sim \delta^{-1/2} \), the following holds

\[
\left( \frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} e^{ix \cdot \xi} a_\xi \right|^p \right)^{1/p} \lesssim \delta^{\frac{n-1}{p-1}} \| a_\xi \|_{l^p(\Lambda)}.
\]  

(\(^**\))

(Here \( |B_R| \) denotes the volume of \( B_R \))

1. Prove the exponent in (*) is necessary by applying (*) to

\[
f(x) = 1_{B(N, \delta) \cap S^{n-1}}(x).
\]

Here \( N \in S^{n-1} \) is a point and \( B(N, \delta) \) is a ball of radius \( \delta \) centered at \( N \).

(Estimate \( f \) by \( 1_{Q_\delta} \) where \( Q_\delta \) is a \( \delta \times \cdots \times \delta \times \delta^2 \) rectangle; the Fourier transform of \( 1_{Q_\delta} \) is “essentially supported” on a \( \delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2} \) rectangle).

1
2. For \( p \geq \frac{2(n+1)}{n-1} \), conjugate \( q \) and some \( C > 0 \), show the equivalences:

\[
\begin{align*}
\text{(a)} & \quad \| \hat{f} \sigma \|_{L^p} \leq C \| f \|_{L^2(\sigma)}. \\
\text{(b)} & \quad \| \hat{f} \|_{L^2(\sigma)} \leq C \| f \|_{L^q}. \\
\text{(c)} & \quad \| \hat{\sigma} \ast f \|_{L^p} \leq C^2 \| f \|_{L^q}.
\end{align*}
\]

3. Show the equivalence between (*) and (**).

Remarks:
The inequality (**), although discrete is richer than the continuous counterpart (*). For example, looking at higher spatial scales \( R \sim \delta^{-1} \), it is possible to obtain an improvement on the \( \delta \) exponent,

\[
\left( \frac{1}{|B_R|} \int_{B_R} \left| \sum_{\xi \in \Lambda} e^{ix \cdot \xi} a_\xi \right|^p \right)^{1/p} \lesssim \delta^{\frac{n+1}{2p}} \frac{n+1}{4} \| a_\xi \|_{L^2(\Lambda)}.
\]

I found this fact, that a discretized version can be richer than its continuous counterpart, quite interesting and wanted to share it with you.

This improvement (\( \diamond \)) is a direct application of the \( l^2 \) decoupling inequality recently proven in [BD]. For this and numerous other applications, see [BD].

References

[S] Stein, E. “Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals”.