Problem Solving Seminar

AMS Grad Student Chapter, University of Rochester

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Problem 1 (+[3] contributed by Lucas). Let $S$ be a smallest class of functions $\mathbb{R}^2 \to \mathbb{R}$ (possibly with singularities) such that

- $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are in $S$.
- $f, g \in S \implies f + g, f - g \in S$
- $f \in S \implies 1/f \in S$

Does the map $(x, y) \mapsto 2019$ belong to $S$?

Problem 2 (++, contributed by Firdavs). Consider the vector space $V$,

$$V = \{(a_n)_{n=1}^\infty : a_n \in \mathbb{F}\}$$

the space of sequences over the field $\mathbb{F}$ with addition and scalar multiplication given by component-wise operations. Show that the Hamel basis of $V$ over $\mathbb{F}$ is not countable. ☐

Problem 3 (????). Is there a non-degenerate notion of a topological manifold over a finite field $\mathbb{F}_p$, or a topological manifold over the algebraic closure $\overline{\mathbb{F}}_p$?

Problem 4 (??, IMO 2009†). Let $a_1, \ldots, a_n$ be distinct positive integers and let $M$ be a set of $n - 1$ positive integers not containing $s = a_1 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at point 0 and making $n$ jumps to the right with lengths $a_1, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in $M$.

*+ indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness. ? indicates I don’t know the solution, and the number of ?s indicates how hard I think the solution probably is.
†solved in a mini-polymath [Tao]
References


Hints

1. Probably obvious, but diagonalize (in both obvious senses of the word).