Problem 1 (++, Putnam 1971). Let $c$ be a real number such that $n^c$ is an integer for every positive integer $n$. Show that $c$ is a non-negative integer.

Problem 2 (? Moscow State University (2013)). Let $x_1, \ldots, x_k \in S^{n-1} = \mathbb{R}^n$ be points on the unit sphere such that $0 \in \text{conv}\{x_1, \ldots, x_k\}$ where $\text{conv}$ represents the convex hull. With the convention that $x_{k+1} = x_1$, show that

$$
\sum_{j=1}^{k} \|x_j - x_{j+1}\| \geq 4
$$

Problem 3 (++, Hat00). For $1 \leq p < \infty$

$$
\ell^p(\mathbb{N}) = \left\{(a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}
$$

Now, let

$$
S^\infty_p = \{x \in \ell^p(\mathbb{N}) : \|x\|_{\ell^p(\mathbb{N})} = 1\}
$$

Show that $S^\infty_p$ is contractible.

Further, let

$$
V_0 = \{(a_n)_{n=1}^{\infty} : \exists N, \forall n \geq N, a_n = 0\}
$$

be the set of sequences that are eventually zero. Show that $V_0 \cap S^\infty_p$ is contractible.

Problem 4 (??, Problem 16 from Nature). Let $R$ be a noncommutative ring with unity, and suppose the elements $x, y \in R$ are such that both $(1 - xy)$ and $(1 - yx)$ are invertible. Show that

$$
(1 + x)(1 - yx)^{-1}(1 + y) = (1 + y)(1 - xy)^{-1}(1 + x)
$$

* + indicates hardness; the more plusses there are, the harder I think the problem is. Conversely, − indicates easiness. ? indicates I don’t know the solution, and the number of ?s indicates how hard I think the solution probably is.

† Translated by Firdavs from their geometry contest

‡ This problem was paraphrased by Brian
References


Hints

1. What happens when you take successive differences?