MTH393A Report
Additive Combinatorics and Incidence Geometry: The Kakeya Problem

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Abstract

Additive Combinatorics is new discipline in mathematics with connections to additive number theory, fourier analysis, graph theory and probability. The field has numerous applications to various other fields, including Incidence Geometry (which focuses on the properties of lines and points in various geometries in a combinatorial sense). We consider the survey of Additive Combinatorics and its applications to Incidence Geometry by Zeev Dvir [1], and present in particular the Kakeya problem from Chapter 4 of [1]. The Kakeya problem deals with the rough notion of "size" of a subset of $\mathbb{R}^n$ or of $\mathbb{F}^n$ (or in general, any geometry with a well-defined notion of lines and direction) which has a "line" in every "direction". We consider the cases of the reals and the finite fields. We examine the state of the art on the problem.

This project was done in conjuction with Vijay Keswani (11799), a fourth year undergraduate in the Dept. of Computer Science and Engineering, who read on the Szemeredi-Trotter theorem on counting incidences between N points and N lines in a given geometry at the same time as when I was reading on the Kakeya problem, and both of us attended each others project presentations. The presentations were also attended by Prof. Shobha Madan from the Dept. of Mathematics and Statistics and Dr. Nitin Saxena and Dr. Rajat Mittal from the Dept. of Computer Science and Engineering. Throughout this report, we follow [1], except where we note otherwise.

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Contents

Notation

Throughout this report, we use the Vinagradov notation

$$f \ll g \text{ and } f \gg g$$

interchangably with the Landau big-oh notation

$$f = \mathcal{O}(g)$$

to mean that there exists a positive constant C such that

$$f \leq Cg$$

The implicit constant C may depend on some quantities (say ϵ, δ etc.). In this case, the quantities may be specified either in writing or as a subscript (say $\ll_\epsilon$ or $\mathcal{O}_\delta$).

We also use the somewhat non-standard notation

$$f \sim g$$

to denote both $f \ll g$ and $f \gg g$ occurring simultaneously.

We will use $\mathbb{F}$ exclusively to denote a finite field with cardinality q.

We will use $\mu(A)$ to denote the Lesbesgue measure of a subset $A \subset \mathbb{R}^n$.

We use the indicator notation

$$1_P = \begin{cases} 1 & \text{if } P \text{ holds} \\ 0 & \text{if } P \text{ does not hold} \end{cases}$$

A $\star$ will be used to denote any theorem which has not been proved in this report.

1 Introduction and Preliminaries

In this reading project, we considered the fields of Additive Combinatorics and Incidence Geometry. In particular, we looked at the Kakeya problem in both the reals as well as the finite fields. In this report, we record the material by me during project presentations, starting with a basic introduction to Additive Combinatorics (we will quote a few results without proof). We then move on to an introduction to the Kakeya problem, and treat the problem first in the finite field case, and then in the reals case, in both cases proving as much as was done in [1], trying to give more context, and giving the state of the art.

We will assume basic familiarity with group theory, finite fields, real projective spaces, affine spaces and real numbers. We will also assume some familiarity with using the Cauchy-Schwarz inequality, which will be a fundamental tool we will use almost everywhere.

1.1 Additive Combinatorics

The field of Additive Combinatorics is a relatively new field which is connected to, and uses ideas from additive number theory, group theory, graph theory and probability. We refer the reader to [2] for an overview of Additive Combinatorics with a specific view towards Computer Science.

In the general setting of Additive Combinatorics, one studies the combinatorial properties of some commutative group G. In particular, suppose $(G, +)$ is the group written in additive notation, and suppose $A, B \subset G$ are arbitrary subsets, then we define the sumset $A + B$ as follows:

$$A + B := \{a + b : a \in A, b \in B\}$$

Note that for a commutative group, this is a commutative set operation. Furthermore, it is an associative set operation.

We can define the difference set $A - B$ in a similar manner (where $a - b = a + (-b)$ and $-b$ is the additive inverse of b in G). We will also use $2A$ to denote $A + A$, $3A$ to denote $A + A + A$ and so on. We will also define the k-dilate as follows:

$$k \cdot A = \{ka : a \in A\}$$

Sometimes we may abuse notation and use kA where we actually mean $k \cdot A$ assuming that the situation is clear from the context. We will also abuse notation to denote the set $\{a\} + A$ as $a + A$.

For typical applications, G will either be the reals or the finite fields, or related groups such as integers, rationals etc. Furthermore, most theorems and applications deal specifically with the case where A and B are finite. We are then interested in the cardinality of the sets and sumsets, and the relationship between them.

In particular, we have the following basic inequality:

Theorem 1.1 (Basic Sumset Inequality)**.** *For the real numbers $\mathbb{R}$, and finite subsets $A, B \subset \mathbb{R}$, we have the following inequality:*

$$|A| + |B| - 1 \leq |A + B| \leq |A||B|$$

further, equality may occur on both sides. Further, if $\mathbb{R}$ is replaced by any arbitrary group, the upper bound still holds.

Proof. The upper bound is trivial. To see this, note that the map $(a, b) \mapsto a + b$ is a map from $A \times B$ to G whose image is exactly $A + B$. Since this map is surjective on $A + B$, we get that

$$|A + B| \leq |A \times B| = |A||B|$$

Further, for $\mathbb{R}$ let $r = |A|$, $s = |B|$ and let $A = \{a_1, \cdots, a_r\}$ and $B = \{b_1, \cdots, b_s\}$. Further, arrange the indices in a manner so that

$$a_1 > a_2 > \cdots > a_r$$

and

$$b_1 > b_2 > \cdots > b_s$$

We thus have that

$$a_1 + b_1 > a_2 + b_1 > \cdots > a_r + b_1$$

and we have

$$a_r + b_1 > a_r + b_2 > \cdots > a_r + b_s$$

and thus, the sequence $a_1+b_1, a_2+b_1, \cdots, a_r+b_1, a_r+b_2, \cdots, a_r+b_s$ consists of $r + s - 1$ distinct elements in $A + B$. Thus, we have that

$$|A+B| \geq \#\{a_1+b_1, a_2+b_1, \cdots, a_r+b_1, a_r+b_2, \cdots, a_r+b_s\} = |A|+|B|-1$$

This gives the inequality. To see that both equalities can occur, take the case where A and B are arithmetic progressions with the same common difference, and the case where $A = \{0, 1, 2, 3, \cdots n\}$ and $B = \{0, n+1, 2n+2, 3n+3, \cdots mn+m\}$.

□

The basic upper bound denoted above is weak but pretty useful in many circumstances.

The properties of subsets under this set operation are very useful in characterizing "structure" in the subsets. For example, if A is a subgroup, we automatically have that $A + A = A$, and thus $|2| = |A|$. In fact, $|2A| = |A|$ implies that A is either a group or a coset of a group. To see this, note that we can assume without loss of generality that $0 \in A$, for if this is not so, we can replace A with $A - a$ for some $a \in A$. Hence, $A = 0 + A \subset A + A = 2A$. Further, $|2A| = |A|$. Hence, we must have $2A = A$, and thus we have that A is finite close subset of G, and hence a closed subgroup of G.

The basic theory of set addition is known by the name of Rusza Calculus. We will now present some basic Rusza Calculus.

1.2 Rusza Calculus

The fundamental result in Rusza Calculus is the triangle inequality, viz.

Theorem 1.2 (Rusza triangle inequality)**.** *Let G be an abelian group, and $A, B, C \subset G$. Then we have the following inequality among cardinalities:*

$$|A||B-C| \leq |A+B||A+C|$$

Proof. For any $x \in B - C$, fix a representation $x = b - c = b(x) - c(x)$ with $b \in B$ and $c \in C$. Now define a map $f : A \times (B - C) \to (A + B) \times (A + C)$ as $f(a, x) = (a + b, a + c)$.

Now, suppose $f(a, x) = f(a', x')$. Thus, $x = b - c = (a + b) - (a + c) = (a' + b') - (a' + c') = b' - c' = x'$. Since we fixed a representation, this implies that $b = b'$ and $c = c'$, and hence $a = a'$.

Thus, f is an injective map. Comparing the cardinalities of the domain and co-domain, the theorem follows.

□

To see why this theorem is called a triangle inequality, we first define the following notion of distance between subsets of a group:

Definition 1.1 (Rusza distance)**.** The *Rusza distance* $d(A, B)$ between two sets $A, B \subset G$ is defined as

$$d(A,B) = \log \frac{|A-B|}{|A|^{1/2}|B|^{1/2}}$$

It is easy to see that this distance is symmetric since $|A - B| = |B - A|$. Further, note that the triangle inequality

$$d(A,C) \leq d(A,B) + d(B,C)$$

can be rewritten, by taking exponentials both sides to

$$\frac{|A-C|}{|A|^{1/2}|C|^{1/2}} \leq \frac{|A-B|}{|A|^{1/2}|B|^{1/2}} \times \frac{|B-C|}{|B|^{1/2}|C|^{1/2}}$$

or, in other words,

$$|A-C||B| \leq |A-B||B-C|$$

This is equivalent to the previous theorem (which can be seen by replacing (A, B, C) in the previous theorem by $(-B, A, C)$).

Thus, the previous theorem is actually equivalent to the statement that the Rusza distance defined above satisfies the triangle inequality.

The Rusza disance is, however, clearly not reflexive in the general case.

The Rusza distance is a very useful tool for proving general inequalities. In particular, it allows us to connect the notion of sets that grow slowly under addition and substraction. For example, we have the following theorem:

Theorem 1.3. *If $|A+A| \leq K|A|$ for some absolute constant K, then we have $|A-A| \leq K^2|A|$. Conversely, $|A-A| \leq K|A|$ implies $|A+A| \leq K^2|A|$.*

Proof. Note that

$$\frac{|A-A|}{|A|} = \exp(d(A, A)) \leq \exp(d(A, -A) + d(-A, A)) = \frac{|A+A|^2}{|A|^2}$$

and that

$$\frac{|A+A|}{|A|} = \exp(d(A, -A)) \leq \exp(d(A, A) + d(-A, -A)) = \frac{|A-A|^2}{|A|^2}$$

Both these inequalities together with the respective hypothesis give the desired conclusion.

□

For a given K, a set satisfying $|A+A| \leq K|A|$ is said to be a set of small doubling. The expectation is that if A has small doubling, then in fact, all possible additions and substractions of A with itself must be small (since there must be inherent structure in A of some sort). The formal result is by Plünneke and Rusza:

Theorem 1.4 (Plünneke-Rusza inequality, $\star$)**.** *Let G be an abelian group, and $A, B \subset G$ be sets of equal size satisfying $|A+B| \leq K|A|$. Then we have*

$$|kA - lA| \leq K^{k+l}|A|$$

We now move on to an introduction to the Kakeya problem.

1.3 The Kakeya Problem

To describe the Kakeya problem, we first need the loose notion of a Kakeya set in a given geometry (with a well-defined notion of "lines" and "directions" over a given field.

Definition 1.2 (Kakeya sets)**.** A Kakeya set is a subset of the underlying space of a geometry which has a line in every direction.

More precisely, we have the following definitions in the special cases of geometries involving finite-dimensional linear spaces over the field of real numbers and over finite fields.

Definition 1.3 (Kakeya sets in $\mathbb{R}^n$)**.** A Kakeya set $K \subset \mathbb{R}^n$ is a compact set such that for every $x \in S^{n-1}$, there exists a $y \in K$ such that

$$l = \{y + tx : t \in [0, 1]\} \subset K$$

Here "line" is defined as a unit line segment (denoted by l), and "direction" is defined in the colloquial sense as a unit vector in $\mathbb{R}^n$ (that is, a point on the unit hypersphere S^{n-1}. Note that in the literature, real Kakeya sets are also called Besicovitch sets.

The fundamental Kakeya problem is to give some meaningful estimate of "size" for a Kakeya set. In particular, we want some sort of moral lower bound on the "size" of this set.

For a compact subset of $\mathbb{R}^n$, one meaningful sense of "size" is that of the Lesbesgue measure or the Jordan-Riemann measure, which is the formal notion of the intuition that we obtain from units of measurement such as length, area and volume. In particular, the Lesbesgue measure in $\mathbb{R}^n$ with the usual topology is the notion of hypervolume that we will normally use.

It so happens, however, that the Lesbesgue measure is not a good notion of size for dealing with Kakeya sets: it was shown by Besicovitch in 1928 [4] that there exists a Kakeya set in $\mathbb{R}^n$ which has Lesbesgue measure equal to zero.

Another useful notion of size of a set is its dimension. For example, in $\mathbb{R}^{\nVDash}$, a point has dimension 0, a line segment has dimension 1, a disc has dimension 2 and a solid sphere has dimension 3, in the "intuitive" sense: after dealing with many inter-related notions of dimension so long, we have an intuitive feel of what the dimension of a particular object should be. To discuss the

Kakeya problem in any meaningful sense with dimension as the notion of size, we will have to rigorously define what we mean by dimension.

One of the various notions of dimension is that of linear independence: with this in mind, it seems that since a Kakeya set should have a line in every direction, it is reasonable to expect that the dimension of the set should in fact be as high as it could probably go - that is, all the way to n. With a suitable notion of "dimension", we thus have the following formulation of the Kakeya problem:

Conjecture 1.1 (Kakeya problem with Dimension)**.** Let $K \subset \mathbb{R}^n$ be a Kakeya set, then we have

$$\dim K = n$$

for some suitable notion of dimension.

We will explore this topic in Section 3.

In the finite field case, the Kakeya set is defined as follows:

Definition 1.4 (Kakeya sets in $\mathbb{F}^n$)**.** A Kakeya set $K \subset \mathbb{F}^n$ is a set such that for every $x \in \mathbb{F}^n - \{0\}$, there exists a $y \in K$ such that

$$l = \{y + tx : t \in \mathbb{F}\} \subset K$$

Clearly, here we no longer have a suitable notion of length, and hence it does not make sense to talk about either a "unit" line segment, nor of "unit" vectors, which results in obvious modifications to the notion of "line" and "direction" for the finite field case.

One clear difference between the real setting and the finite field setting is the fact that unlike reals, we have only one really meaningful notion of size of sets in the finite field setting - cardinality. Since all sets are finite, it makes most sense to work completely with cardinality.

Now, as we will see later, there is a sort of similarity to considering the finite field case with the size of the field q going to infinity, to the real case with balls of radius ϵ covering the Kakeya set as $\epsilon \to 0$. As a matter of fact, using the relation $q \sim 1/\epsilon$ gives a nice heuristic method for translating expected results back and forth between the finite field and real setting.

In any case, we can now state the Kakeya problem for the finite field setting, which was solved completely by Zeev Dvir (the author of the survey we are following) in [5].

Theorem 1.5 (Kakeya problem in Finite Fields)**.** *For any positive integer n, for all finite fields $\mathbb{F}$ of cardinality q we have that for any Kakeya set $K \subset \mathbb{F}^n$, we have*

$$|K| \gg_n |\mathbb{F}|^n = q^n$$

where the implicit constant is independent of the underlying field and only depends on the dimension of the ambient space over the field.

In other words, for any positive integer n, there exists an absolute constant C_n such that for any Kakeya set K in any finite field $\mathbb{F}$ we have that

$$|K| \geq C_n |\mathbb{F}|^n = C_n q^n$$

We will prove this theorem completely in Section 2.

2 The Kakeya Problem over Finite Fields

In this section, we will directly provide a beautiful proof of the best result in [1] regarding the Kakeya problem, due to Zeev Dvir. We will also use [6] as a useful reference for our arguments.

Before we move on to the crux of the proof, we will need some basic facts about projective spaces.

2.1 The Projective Space $\mathbb{PF}^n$

The projective space over a finite field $\mathbb{F}$ is defined pretty similarly to projective spaces over the real numbers $\mathbb{R}$ the n-dimensional projective space is essentially the set of all directions in the $n+1$-dimensional linear space over $\mathbb{F}$. More formally, it is the space obtained by collapsing all points lying on lines passing through the origin into each other. That is,

Definition 2.1 (Projective Space over $\mathbb{F}$)**.** Let $\mathbb{F}^{n+1}$ be the $n+1$ dimensional linear space over $\mathbb{F}$. We define the equivalence relation $\sim_P$ for $x, y \in \mathbb{F}^{n+1} - \{0\}$ as follows:

$x \sim_P y$ if and only if there exists a non-zero $\lambda \in \mathbb{F}^*$ such that $x = \lambda y$. We call the resulting quotient space under this relation as the *projective space of dimension n over* $\mathbb{F}$, denoted as $\mathbb{PF}^n$.

We will call the process of taking the equivalence relations *projectivizing.* Furthermore, all linear maps from $\mathbb{F}^{n+1}$ that remain well-defined after projectivization shall be known as projective maps from $\mathbb{PF}^n$.

Points in $\mathbb{PF}^n$ shall be denoted by the $n+1$ homogenous coordinates (which are unique up to multiplication by a non-zero scalar) $x = (x_0 : x_1 \cdots : x_n)$.

Now note that the n-dimensional affine space $\mathbb{F}^n$ can be embedded into $\mathbb{PF}^n$ by mapping the point $(x_1, \cdots, x_n) \in \mathbb{F}^n$ to $(1 : x_1 : \cdots : x_n) \in \mathbb{PF}^n$, and this map will respect the structure (whereby projective maps will reduce to affine maps for the embedded affine space). Once this embedding has been fixed, the points in $\mathbb{PF}^n$ having $x_0 = 0$, that is, points of the form $(0 : x_1 : \cdots : x_n)$ are known as the *points at infinity.* The set of all these points is then known as the hyperplane at infinity, analogous to real projective case.

Now consider any line l in $\mathbb{F}^n$ say

$$a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

It is easy to see that after projectivizing, $x_0 = 1$, so by homogenizing the line will now become

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$$

This "line" is now completed to include points with $x_0 = 0$. It is easy to see that affine points on this completed line are the same as those on the unprojectivized line.

We now take a moment to note that later in the report, we will be modifying an elementary method used to bound the size of sets in algebraic combinatorics which is called the *polynomial method* so that it can be applied to projective spaces. See [3] for a recent survey on the applications of this methods in various fields.

To apply the polynomial method on projective spaces over finite fields, we will first need a suitable notion of "polynomial" over a projective space. In particular, we need the notion of "zero" to be well-defined for these polynomials.

It is not very difficult to see that the notion of polynomials over $\mathbb{PF}^n$ would actually be set of all homogenous polynomials in $\mathbb{F}[x_0, x_1, \cdots, x_n]$. That is, all polynomials whose monomials have the same degree. It is easy to see that polynomial will remain well-defined as a function after projectivization, since we can easily note that being homogenous is equivalent to satisfying

$$f(ax_0, ax_1, \cdots, ax_n) = a^d f(x_0, x_1, \cdots, x_n)$$

where $d = \deg f$, and a is a non-zero element of $\mathbb{F}$. Hence, $x \sim_P y$ implies that $f(x) \sim_P f(y)$.

Due to this, we will denote the set of homogenous polynomials over the $n+1$ variables $(x_0, x_1, \cdots, x_n)$ as $\mathbb{PF}[x_0 : x_1 : \cdots : x_n]$.

In particular, note that the standard embedding of $\mathbb{F}^n$ into $\mathbb{PF}^n$ actually gives an embedding of $\mathbb{F}[x_1, \cdots, x_n]$ into $\mathbb{PF}[x_0 : x_1 : \cdots : x_n]$ as follows: for any polynomial $f \in \mathbb{F}[x_1, \cdots, x_n]$ of degree d, multiply every monomial in f with a power x_0^r such that the degree of the monomial becomes equal to d

(and makes the result a homogenous polynomial in the $n+1$ variables, say f^h). In other words, we consider the map $f \mapsto f^h$ given by

$$f^h(x_0, x_1, \cdots, x_n) := x_0^d f(x_1/x_0, \cdots, x_n/x_0)$$

To see that this map is injective (and thus an embedding) note that substituting $x_0 = 1$ in f^h gives back f. Also note that this also demonstrates that the given embedding is consistent with the embedding of $\mathbb{F}^n$ into $\mathbb{PF}^n$.

Finally, note that setting $x_0 = 0$ gives the restriction of f^h to the hyperplane at infinity, and is in fact, equal to the homogenous part of highest degree of f.

2.2 Proof of the Finite Field Problem

We now present the proof of the conjecture.

Before we move on to the proof, we state the following lemma

Lemma 2.1 (Schwartz-Zippel Theorem,$\star$)**.** *Let $f \in \mathbb{F}[x_1, \cdots, x_n]$ be a polynomial of degree d. Then there are at most dq^{n-1} points in $\mathbb{F}_n$.*

The above theorem is easily proved by induction. In any case, the important information in this is that for every non-vanishing polynomial with degree less than the size of the field, there is at least one point in $\mathbb{F}^n$ on which the polynomial does not vanish. This fact is much easier to prove by induction than the Schwartz-Zippel Theorem.

We now begin the proof of the Kakeya problem in finite fields. Following [6], we prove the following two lemmas.

Lemma 2.2. *Let $E \subset \mathbb{F}^n$ be a set such that $|E| < \binom{n+d}{d}$ for some $d \geq 0$. Then there exists a non-zero polynomial $f \in \mathbb{F}[x_1, \cdots, x_n]$ with degree at most d which vanishes on E.*

Proof. Consider $V \subset \mathbb{F}[x_1, \cdots, x_n]$, the set of all polynomials of degree at most d, which forms a vector space over $\mathbb{F}$. It is easy to see that the set of all monomials with degree at most d forms a basis for this vector space. Further, note that the number of such monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ is exactly the same as the number of non-negative integral solutions of the inequality

$$k_1 + k_2 + \cdots k_n \leq d$$

A little combinatorial argument will give that this equation has $\binom{n+d}{d}$ solutions.

Now note that vector space (say W) of $\mathbb{F}$-valued functions over E, that is the set of all functions $f : E \to \mathbb{F}$, taken as a vector space over $\mathbb{F}$ has the basis given by the set of all f_e for $e \in E$ given by

$$f_e(x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{if } x \neq e \end{cases}$$

which has cardinality $|E|$. Hence, this vector space has dimension equal to $|E|$.

Now note that there exists the natural linear map from V to W given by $f \mapsto (f(x))_{x \in E}$. Further, since,

$$\dim W = |E| < \binom{n+d}{d} = \dim V$$

this map cannot be injective (as othen the image of V under the map will have dimension as a subspace greater than the ambient space). Thus, there exists at least two polynomials which have the same value on all $x \in E$. In particular, the difference of these two polynomials is identically zero at all points in E. Hence, we are done.

□

Morally speaking, this lemma states that if any set is small enough, there is at least one low-degree polynomial which vanishes on it entirely. Thus, if there is a set on which no low-degree polynomial vanishes, it must follow that that set must be large.

In other words, we get the contrapositive which allows us to bound below the size of a set by showing that no polynomial of low degree vanishes on it. Explicitly, if no polynomial of degree less than d vanishes on a set E, then we must have

$$|E| \geq \binom{n+d}{d} = \binom{n+d}{n}$$

We can now combine this lemma with the next lemma to get our result.

Lemma 2.3. *If $K \subset \mathbb{F}^n$ is a Kakeya set, and f is a polynomial of degree at most $q-1$ which vanishes on K, then $f \equiv 0$.*

Proof. Let f be a polynomial which vanishes on the Kakeya set K. Consider the canonical embedding as described earlier of $\mathbb{F}^n$ into $\mathbb{PF}^n$. The Kakeya set K will now be mapped to some set $K' \subset \mathbb{PF}^n$. Further, the polynomial f will be mapped to f^h, the homogenization of f.

Now in any direction we have a line in K. Thus, in $\mathbb{PF}^n$, for every line passing through a point at infinity, we shall have q affine points lying on that line which are also in K'. Thus the restriction of f^h on the line (say parametrized by t so that we are looking at something like $f^h(a+tv)$ for fixed v, and thereby a consequent a. This is a polynomial in t with degree less than the degree of f, which is itself less that $q-1$. However, it vanishes on q points. Thus, this function must be zero everywhere on the line.

This implies that at all points at infinity, f^h vanishes. That is,

$$f^h(0, x_1, \cdots, x_n) = 0$$

for all $(x_1, \cdots, x_n) \in \mathbb{F}^n$. By the Schwartz-Zippel lemma, since the degree is less than q, it follows immediately that $f^h(0, x_1, \cdots, x_n)$ is the zero polynomial.

However, $f^h(0, x)$ is the homogenous part of f with highest degree. It automatically follows that f must be zero, which gives a contradiction.

$\square$

Combining the lemmata above, we clearly get that for a Kakeya set K, we have

$$|K| \geq \binom{n+(q-1)}{n} \geq \frac{q^{n-1}(q-1)}{n!} \gg_n q^n$$

which solves the Kakeya problem in finite fields with an implicit constant of the order of $1/n!$.

3 The Kakeya Problem over Reals

In this section, we will survey the state of the art for the Kakeya problem over the reals. We will solve completely the only case of the Kakeya problem over the reals which has been solved (viz. the case $n = 2$). We will define a suitable notion of dimension and elucidate some of its properties, following which we will prove two lower bounds on the dimension of a Kakeya set (morally the Kakeya problem is proving that the lower bound on the dimension of a Kakeya set is the same as the upper bound, which is n), the original idea behind which is from papers by Bourgain, Katz and Tao (in particular [7] and [8]). We will also describe a general recipe to create a Bourgain type lower bound on the dimension using a problem which is purely from Additive Combinatorics and does not involve Incidence Geometry or the Kakeya problem in any ostensible way.

Before we move any further, we will examine the notion of "size" we are going to use in this circumstance, which is the "box dimension", which is also known as the Minkowski dimension (or more properly, the upper Minkowski dimension).

3.1 Upper Minkowski Dimension

For the rest of this report, we will use the word dimension to exclusively mean the upper Minkowski dimension which is defined as follows:

Definition 3.1 (Upper Minkowski Dimension)**.** For any bounded set K, suppose $B_\epsilon(K)$ is the minimal number of balls of radius ϵ using which K can be completely covered (this number exists since K is completely contained in a compact set). The dimension of K, or more properly, the *Upper Minkowski Dimension* of K is given by

$$\dim K = \limsup_{\epsilon \to 0} \frac{B_\epsilon(K)}{\log(1/\epsilon)}$$

Loosely speaking, if $\dim K \leq d$, then something like $\sim (1/\epsilon)^d$ balls of radius ϵ will be needed to cover K.

We would now like to demonstrate that this notion of dimension shall agree with out intuitive notion of dimension in most expected cases. For this purpose we consider the following properties of the defined dimension.

Property 3.1 (Monotonicity). *Suppose $L \subset K \subset \mathbb{R}^n$ such that K and L are both bounded. Then,*

$$\dim L \leq \dim K$$

Proof. This is trivial. Note that since $L \subset K$, any covering of K will also cover L. Thus, the number of balls needed to cover L is less than the number of balls needed to cover K at all times.

Thus,

$$B_\epsilon(L) \leq B_\epsilon(K)$$

Thus, by the definition of dimension,

$$\dim L \leq \dim K$$

□

Property 3.2 (Upper Bound). *Suppose $K \subset \mathbb{R}^n$ is a bounded set. Then,*

$$\dim K \leq n$$

Proof. Note that since K is bounded, it must be contained in some large enough ball, B. Thus, we have by the previous property

$$\dim K \leq \dim B$$

As we shall show later, any bounded subset of $\mathbb{R}^n$ having positive Lesbesgue measure has dimension n, including, in particular, all balls. Thus,

$$\dim K \leq \dim B = n$$

□

Property 3.3 (Dimension of sets of non-zero measure)**.** *Let $K \subset \mathbb{R}^n$ be a bounded subset having positive Lesbesgue measure $\mu(K) > 0$. Then,*

$$\dim K = n$$

Proof. Consider the cover of $B_\epsilon(K)$ balls of radius ϵ covering K. Clearly, the total Lesbesque measure of the balls, μ_{total} cannot exceed the number of balls times the Lesbesgue measure of each ball. Further, we can choose a ball S which is so big that it contains the set K along with the cover of ϵ-radius balls for sufficiently small ϵ (say $\epsilon < 1$). This can easily be done by taking the radius of the ball to be many times the diameter of K plus 1. Thus,

$$\mu_{total} \ll B_\epsilon(K) \times \epsilon^n \ll \mu(S)$$

Since K is totally covered, $\mu(K) < \mu_{total}$, and hence,

$$\mu(S) \gg B_\epsilon(K) \gg \frac{\mu(K)}{\epsilon^n} \gg_K \frac{1}{\epsilon^n}$$

After taking logarithms, the implicit constant will become an additive constant, and will disappear when the limsup is taken. Thus we will get that

$$n \geq \dim K \geq n$$

□

Property 3.4 (Affine Dimension)**.** *Let A be an affine subspace of $\mathbb{R}^n$ having affine dimension k. Then, for any ball B having a non-trivial intersection with A we have that*

$$\dim A \cap B = k$$

Proof. To see this, note that if we map the affine subspace injectively onto $\mathbb{R}^k$, the image of $A \cap B$ will go to a ball in $\mathbb{R}^k$ (which thus has dimension k). It is relatively easy to see that Minkowski dimension is preserved under injective continuous maps, and thus we are done.

□

This last property essentially shows that the "affine" concept of dimension gives the same value of dimension as the upper Minkowski dimension.

Property 3.5 (Tensoring)**.** *Let K be a bounded subset of $\mathbb{R}^n$. It then follows that K^t is a bounded subset of $\mathbb{R}^{nt}$ under some suitable canonical maps. Thus, we have*

$$\dim K^t = t \dim K$$

Proof. To see this, it is sufficient to note that the direct products of all the balls covering K with each other with t possibilities will give a ball covering for K^t. It is not difficult to see that

$$B_\epsilon(K^t) \sim B_\epsilon(K)^t$$

Taking logarithms, dividing and taking limsup gives the desired property.

□

At this point, we are going to make a couple of claims about the Minkowski dimension whose proof we will only sketch, rather than giving the details. The truth of these statements can be easily intuited, and hence these claims will be called "touchy-feely claims". We will use these claims as axiomatic in the rest of the report, when necessary.

Touchy-Feely Claim 3.1 (Grid-Points)**.** *Consider the lattice given by $\epsilon\mathbb{Z}^n$ as a subset of $\mathbb{R}^n$ (with the same axes and origin). We define $G_\epsilon^\alpha(K)$ to be the number of grid-points from the lattice that are at a distance at most $\alpha\sqrt{n}$ from K.*

Then, for sufficiently large α, we have

$$B_\epsilon(K) \sim_{\alpha,n} G_\epsilon^\alpha(K)$$

In particular, in the definition of dimension, B can be replaced by G^α.

Proof Sketch. To see this, note that if we take a ball with a center at every such grid-point, then the set of all such balls would cover K completely. In other words, we have $G_\epsilon^\alpha(K)$ balls covering K. Thus, by definition

$$G_\epsilon^\alpha(K) \gg B_\epsilon(K)$$

To see the other direction, note that for a fixed α and n, there is a maximum number of grid-points $C_{\alpha,n}$ that is at a distance at most $\alpha\sqrt{n}$ away from a ball of radius ϵ. Thus, the number of grid-points at a distance at most $\alpha\sqrt{n}$ away from the cover shall be at most $C_{\alpha,n}G_\epsilon^\alpha(K)$, and this will be greater than the number of grid-points at a distance at most $\alpha\sqrt{n}$ away from K. Hence, we see that

$$G_\epsilon^\alpha(K) \ll B_\epsilon(K)$$

Combining the two, we get the claim.

□

The touchy-feely part of this claim is the usage of α without justification. This can be made more rigorous. Also, note that this touchy-feely claim essentially says that we can ignore perturbations in α as long as it is sufficiently large.

We now come to the second touchy-feely claim that we are not going to prove completely, which is

Touchy-Feely Claim 3.2 (Heuristic for G_ϵ^α)**.** *For large enough α, and small enough ϵ, if K is a bounded set such that $\mu(K) > 0$, then we have that*

$$G_\epsilon^\alpha(K) \sim_{\alpha,n} \frac{\mu(K)}{\epsilon^n}$$

We will not prove this claim. However, note that a modification of the proof of Property 3.3 will suffice for this purpose. Note also that this heuristic, while useful, breaks down for sets of measure zero (that is, $\mu(K) = 0$).

Now that we have some amount of familiarity with the Minkowski dimension, we can move on to some results.

3.2 The case $n = 2$

For the plane $\mathbb{R}^2$, Kakeya's conjecture has been established in [9]. We will prove this case now.

Theorem 3.1 (Kakeya's problem in $\mathbb{R}^2$)**.** *Let* $K \subset \mathbb{R}^2$ *be a Kakeya set. Then,* $\dim K = 2$*.*

Proof. Let K' be the ϵ-neighbourhood of K (that is, the set of all points a distance at most ϵ away from K.

It is easy to see that

$$G_\epsilon^\alpha(K') \sim G_\epsilon^{\alpha+1}(K) \sim G_\epsilon^\alpha(K)$$

for large enough α. Thus, clearly

$$\dim K' = \dim K$$

Let l_j be the unit line segment in K in the direction $(\cos(\epsilon j\pi/2), \sin(\epsilon j\pi/2)$. Thus, we can take a tube T_j in that direction in K'.

Note that $\mu(T_j) \sim \epsilon$. Further, it can be seen by an easy diagram that

$$\mu(T_i \cap T_j) \sim \frac{\epsilon^2}{\sin(\epsilon|i-j|\pi/2)}$$

for $i \neq j$. Thus using the fact that $\sin x \gg x$, we can see that

$$\mu(T_i \cap T_j) \ll \frac{\epsilon}{|i-j|}$$

Now, using the touchy-feely claim, we have a good handle on what G_ϵ is. In particular, note that there are $\sim (1/\epsilon)$ tubes.

Thus,

$$\frac{1}{\epsilon^2} \sim \sum_j \frac{1}{\epsilon} \sim \sum_j G_\epsilon(T_j)$$

Using indicator functions, we get

$$\sum_{x \in G_\epsilon(K')} \sum_j 1_{x \in T_j}$$

Applying Cauchy-Schwarz on the outer sum,

$$\ll (G_\epsilon(K'))^{1/2} \left(\sum_{x \in G_\epsilon(K')} \left(\sum_j 1_{x \in T_j} \right)^2 \right)$$

Now, changing the square summation into two different summations over different indices, and then interchanging the summation, we see that the bracketed term becomes

$$\sim \left(\sum_{i,j} G_\epsilon(T_i \cap T_j) \right)^{1/2}$$

$i = j$ gives the diagonal term $\sum_j G_\epsilon(T_j) \sim 1/\epsilon^2$, while for the non-diagonal terms we can use our earlier approximation to get

$$\ll \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \sum_{i \neq j} \frac{1}{|i-j|} \right)^{1/2}$$

Here the sum runs until $\sim 1/\epsilon$. Noting that the inner sum has each summand appearing at most finitely many times, and goes up to $1/\epsilon$, we see that the inside can be approximated by

$$\frac{\log(1/\epsilon)}{\epsilon^2}$$

Putting all of this together gives us the bound

$$G_\epsilon(K') \gg \frac{1}{\epsilon^2 \log(1/\epsilon)}$$

Taking logarithms, dividing and taking limsup gives us the claim that

$$\dim K = 2$$

□

We now move on to the $n/2$ bound.

3.3 The $n/2$ bound

In this subsection, we will prove a lower bound originally by Bourgain to the dimension of a Kakeya set in an arbitrary dimension. The proof is somewhat similar in spirit to the previous proof.

We first explicitly state the conjecture

Conjecture 3.1 ($\mathsf{Kakeya}(\beta)$)**.** We say that $\mathsf{Kakeya}(\beta)$ holds over $\mathbb{R}^n$, if for any Kakeya set $K \subset \mathbb{R}^n$, we have the following bound

$$\dim K \geq \frac{n}{\beta}$$

Thus, the full Kakeya conjecture amounts to showing that $\mathsf{Kakeya}(1)$ holds over $\mathbb{R}^n$ for all n.

We will deduce $\mathsf{Kakeya}(2)$ by applying the "tensoring" trick to the following bound:

Theorem 3.2 ($(n-1)/2$ bound)**.** *If $K \subset \mathbb{R}^n$ is a Kakeya set, then*

$$\dim K \geq \frac{n-1}{2}$$

Proof. Our starting point is a notion called "ϵ-separated directions". Loosely speaking, a set of directions $\Omega \subset S^{n-1}$ is said to be ϵ-separated, if any vectors in Ω have distance $\gg \epsilon$ between them, where the implicit constant is absolute for Ω. It is not difficult to see that it is reasonable to expect that we can find an ϵ-separated set Ω such that $|\Omega| \sim (1/\epsilon)^{n-1}$ whereby we are trying to make Ω as uniformly distributed on the surface of S^{n-1} as possible.

Now, since K is a Kakeya set, for every direction $w \in \Omega$, we have a line l_w in K with (say) end-points a_w and b_w. Thus

$$l_w = \{a_w + tw : t \in [0, 1]\} \subset K$$

and $b_w = a_w + w$.

Now consider the grid $\epsilon\mathbb{Z}^n$, and suppose that a'_w and b'_w are the closest grid-points respectively to a_w and b_w. Let l'_w be the line between these grid-points and let w' be the direction of this line. It is easy to see that since Ω

was ϵ-separated, this process cannot collapse too many directions into each other (in fact, no more than 2 in this case). Thus, if we take

$$\Omega' = \{w' : w \in \Omega\}$$

we have the lower bound $|\Omega'| \gg |\Omega|$.

Further, take

$$A = \{a'_w : w \in \Omega\}$$

and

$$B = \{b'_w : w \in \Omega\}$$

Thus, clearly since $w' = b'_w - a'_w$, $\Omega' \subset B - A$.

Thus,

$$|\Omega'| \leq |B - A|$$

We also have from the beginning

$$|B - A| \leq |A||B|$$

Further since A and B are both subsets of the grid-points closest to K, we have that

$$|A|, |B| \leq G^{\alpha}_{\epsilon}(K)$$

Combining all these inequalities, we see that

$$(1/\epsilon)^{n-1} \ll |\Omega| \ll |\Omega'| \ll |B - A| \ll |A||B| \ll G^{\alpha}_{\epsilon}(K)^2$$

Thus, by applying the definition,

$$\dim K \geq \frac{n-1}{2}$$

□

To go from this theorem to $\mathsf{Kakeya}(2)$ we use what is known as the "tensoring" trick. Note that for a Kakeya set $K \subset \mathbb{R}^n$, $K^t \subset \mathbb{R}^{nt}$ is also a Kakeya set. Hence, by applying the above to that, we see that

$$t \dim K = \dim K^t \geq \frac{nt-1}{2}$$

Dividing throughout by t and taking the limit $t \to \infty$, we see that

$$\dim K \geq \frac{n}{2}$$

giving $\mathsf{Kakeya}(2)$.

3.4 The Combinatorial Reduction

Now, as promised before, we examine a problem in Additive Combinatorics, to which $\mathsf{Kakeya}(\beta)$ can be reduced to. Using this reduction, we can then prove, following [1], a slightly stronger lower bound of $4n/7$.

Conjecture 3.2 ($\mathsf{SD}(R,\beta)$)**.** For any positive real number β, and some $R \subset \mathbb{N}$, we say $\mathsf{SD}(R,\beta)$ holds over an abelian group G, if the following holds:

For any subsets $A, B \subset G$ such that $|A|, |B| \leq N$, and for any $\Gamma \subset A \times B$, suppose that for all $r \in R$, we have that

$$|\{a + rb : (a,b) \in \Gamma\}| \leq N$$

Then, we have that

$$|\{a - b : (a,b) \in \Gamma\}| \leq N^{\beta}$$

Using a variant of the argument in the previous subsection, we can now prove the following theorem:

Theorem 3.3 (Reduction)**.** *Suppose* $\mathsf{SD}(R,\beta)$ *holds over* $\mathbb{R}^n$ *with* $R = \{1, 2, \cdots, r\}$*, then* $\mathsf{Kakeya}(\beta)$ *holds over* $\mathbb{R}^n$*.*

Proof. Let K be a Kakeya set.

Fix a sufficiently small ϵ and a sufficiently large α, and define $N = G_\epsilon^\alpha(K)$.

As before, consider an ϵ-separated set of directions $\Omega \subset S^{n-1}$. As before for $w \in \Omega$ we define a_w, b_w, a'_w, b'_w and w'.

However, unlike last time, we make the following change: we move the point (a_w, b_w) to the point (a'_w, b'_w) so that for any $r \in R$ we have that the combination $a'_w + jb'_w$. Note that even though R may be large, it is still finite, and thus this process would require a shifting of $\mathcal{O}(\epsilon)$ of a_w and b_w at most. This process would require some amount of Chinese Remaindering to move the grid-points around in a manner so that we can ensure that for all $w \in \Omega$ and all $j \in R$, we have that $a'_w + jb'_w$ is on a grid-point.

Since Ω started out as ϵ-separated, changin points by $\mathcal{O}(\epsilon)$ distances will not collapse it overtly. Thus we shall have

$$|\Omega'| \gg |\Omega| \gg (1/\epsilon)^{n-1}$$

as before.

Also, note that A, B and each of the following sets, for a fixed j in R,

$$\{a'_w + jb'_w : w \in \Omega\}$$

has cardinality $\ll N$. Thus, if $\mathsf{SD}(R, \beta)$ holds over $\mathbb{R}^n$, we can take A and B as given, and take Γ such that $(a, b) \in \Gamma$ if and only if $a = a'_w$ and $b = b'_w$ with some $w' \in \Omega'$ (that is, we index by Ω'). Thus, by the hypotheses, we can conclude that

$$|\Omega'| = |\{a'_w - b'_w : w' \in \Omega'\}| \ll N^\beta$$

Since we have the lower bound of $(1/\epsilon^{n-1})$ on $|\Omega'|$, we get that

$$G_\epsilon^\alpha(K) = N \geq (1/\epsilon)^{\frac{n-1}{\beta}}$$

Applying the definition of dimension, we get

$$\dim K \geq \frac{n-1}{\beta}$$

Applying the tensoring trick, we get

$$\dim K \geq \frac{n}{\beta}$$

establishing $\mathsf{Kakeya}(\beta)$.

$\square$

Thus, we see that the Kakeya problem reduces to demonstrating $\mathsf{SD}(R, 1)$ for some set $R \subset \mathbb{N}$.

3.5 The $4n/7$ bound

We will now show that $\mathsf{SD}(1, 2, 7/4)$ holds over any abelian group. This will give $\mathsf{Kakeya}(7/4)$, thereby giving a lower bound of $4n/7$.

We first state and prove a useful lemma that is essentially just Cauchy-Schwarz.

Lemma 3.1. *Let W and Z be finite sets, and let $f : W \to Z$ be any map. Then,*

$$|\{(u, v) \in W^2 : f(u) = f(v)\}| \geq \frac{|W|^2}{|\mathrm{Im}(f)|}$$

Proof. First note that we can replace $\mathrm{Im}(f)$ with Z by changing the co-domain. Thus, we note that

$$\sum_{u,v \in W} 1_{f(u)=f(v)} = \sum_{u,v \in W} \sum_{z \in Z} 1_{f(u)=z} 1_{f(v)=z}$$

Interchanging the summations, and noting that the sums over u and v are independent, we can convert the double sum into the square of a single sum. Thus, we get

$$\sum_{z \in Z} \left(\sum_{u \in W} 1_{f(u)=z} \right)^2$$

Applying Cauchy-Schwarz

$$\geq \frac{1}{|Z|} \times \left(\sum_{z \in Z} \sum_{w \in W} 1_{f(u)=z} \right)^2 = \frac{|W|^2}{Z}$$

proving the lemma.

□

We will now interpret the triple (A, B, Γ) loosely as a bipartite graph G with vertex set $V = A \cup B$, and edge set $E = \{a, b : (a, b) \in \Gamma\}$.

Further, we note that we can assume without loss of generality that the size of the set $\{a - b : (a, b) \in \Gamma\}$ is exactly the same as the size of Γ by ensuring that each difference $a - b$ is unique. This can be done since doing so only removes edges, and decreases the bound N for the other sets in the definition of $\mathsf{SD}(R, \beta)$. Thus, we now need to bound $|\Gamma|$ only.

We now define the notion of a "gadget", which is basically just a formal subgraph satisfying certain constraints.

Definition 3.2 (Gadget)**.** A *gadget* is a 4-tuple $G = (V_A, V_B, E, C)$ where $V_A = \{a_1, \cdots, a_s\}$ and $V_B = \{b_1, \cdots, b_r\}$ are sets of formal variables, $E \subset V_A \times V_B$, and C is a set of constraints on the formal variables of the form

$$a_i + rb_j = a_{i'} + r'b_{j'}$$

with $i, i' \in \{1, \cdots, s\}$, $j, j' \in \{1, \cdots, r\}$ and r, r' are integers.

We say a gadget occurs in (A, B, Γ) if there exist injective maps from V_A to A, V_B to B, such that the image of E occurs in the edge set of (A, B, Γ), and such that the constraints in C are satisfied by the values substituted for the formal variables.

Thus, the occurrence of the gadget is a slightly stronger statement than claiming that G is a subgraph of (A, B, Γ).

The proof method shall be the following:

First, we obtain a lower bound on the number of times G appears in (A, B, Γ) in terms of $|\Gamma|$ by using the Cauchy-Schwarz lemma

Second, we obtain an upper bound in terms of N by encoding a gadget with less information.

Before going on to that, let us define our gadget

Definition 3.3 ($G_{4/7}$)**.** The gadget $G_{4/7}$ is given by

$$V_A = \{a_1, a_2 = a_3\}$$

$$V_B = \{b_1, b_2, b_3\}$$

$$E = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3)\}$$

$$C = \{a_1 + 2b_1 = a_3 + 2b_3\}$$

It is clear that the gadget is essentially a path of length three from b_1 to b_3 (that is b_1, a_1, b_2, a_2, b_3) such that the first and the last edge satisfy the constraint that is specified.

We will denote the number of times a $G_{4/7}$ occurs in (A, B, Γ) by T.

We will now first lower bound T.

We have that $|A|, |B| \leq N$ and that for $r = 1, 2$,

$$|\{a + rb : (a, b) \in \Gamma\}| \leq N$$

Now, consider the projection map p from $A \times B$ to A. Further, let

$$M = \{((a, b), (a', b')) \in \Gamma^2 : a = a'\}$$

Thus, clearly, by the lemma stated in the beginnning of this subsection on p, we have that

$$|M| \geq \frac{|\Gamma|^2}{|A|}$$

Now, since $|A| \leq N$, we get

$$|M| \geq \frac{|\Gamma|^2}{N}$$

Now, let $g : M \to G^3$ be given as

$$g((a,b),(a',b')) = (b', a+2b)$$

It is easy to see that every collision of T gives a distinct occurrence of $G_{4/7}$ in (A, B, Γ). Thus,

$$T \geq \frac{|M|^2}{\mathrm{Im}(g)}$$

Now, $\mathrm{Im}(g)$ is contained in a subset of size N^2, and hence

$$T \geq \frac{|M|^2}{N^2} \geq \frac{|\Gamma|^4}{N^2}$$

Now, to get the upper bound on T, we need to note that any given occurrence of a gadget has only so many free variables. In particular, specifying a small set of information about the occurrence of a gadget can uniquely specify the occurrence of the gadget (if such an occurrence even exists). This can be done by using the various constraints.

Let $G' := (a_1, a_2, b_1, b_2, b_3)$ be an occurrence of $G_{4/7}$ in (A, B, Γ). We now show that the triple $(b_3, a_1 + b_2, a_1 + b_1)$ completely specifies the occurrence, if it exists. Since each element lies in a set of size at most N (by hypothesis), it follows that there are at most N^3 such gadgets (that is, $T \leq N^3$.

The proof is as follows:

We know that $a_1 + 2b_1 = a_3 + 2b_3 = a_2 + 2b_3$. Hence,

$$a_2 - b_1 = (a_1 + b_1) - 2b_3$$

All quantities on the right are known, and hence the quantity on the left is known.

Now we compute

$$b_1 - b_2 = (a_1 + b_1) - (a_1 + b_2)$$

Finally, we get

$$a_2 - b_2 = (a_2 - b_1) - (b_2 - b_1)$$

Once we know the difference between a_2 and b_2, we know a_2 and b_2 themselves. Using that, we can get b_1, b_3 and a_1 trivially.

Thus, we have shown that

$$\frac{|\Gamma|^4}{N^4} \leq T \leq N^3$$

Thus,

$$|\Gamma| \leq N^{7/4}$$

establishing $\mathsf{SD}(1, 2, 7/4)$, and thus, $\mathsf{Kakeya}(7/4)$.

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