Moments of the Hurwitz zeta function on the critical line

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(partly joint with Winston Heap and Trevor Wooley)

Overview of the Talk

The zeta functions of Hurwitz and Riemann

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for $\sigma > 1$. This is the shifted integer analogue for the (usual) zeta function of Riemann, $\zeta(s) = \zeta(s, 1)$, given by

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s},$$

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- They both satisfy a "functional equation".

The functional equation of $\zeta(s, \alpha)$

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These can both be viewed as manifestations of the Poisson summation formula.

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- The same is true in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$ for rational shifts (Voronin, 1976) and transcendental shifts (Gonek, 1979). This is open (afaik) for algebraic irrationals!

Moments of $\zeta(s)$

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$$\zeta(\tfrac{1}{2} + it) \ll_\epsilon |t|^\epsilon \iff M_k(T) \ll_{k,\epsilon} T^{1+\epsilon},$$

where the left hand side here is the Lindelöf hypothesis.

Known results about $M_k(T)$

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Precise conjectural values of c_k are now known due to (Keating–Snaith, 2000) via an analogy with random matrix theory.

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We will heuristically justify this expectation for rational α.

Things are more complicated for irrational α – we will return to this later.

Known results about $M_k(T;\alpha)$

The classical mean-square methods for $\zeta(s)$ apply also to $\zeta(s,\alpha)$. (Rane, 1980) showed that uniformly for all $0 < \alpha \leqslant 1$,

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$$= T\log T + B(\alpha)T - \frac{1}{\alpha} + O\left(\frac{T^{1/2}\log T}{\alpha^{1/2}}\right),$$

for an explicit constant $B(\alpha)$.

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for an explicit constant $B(\alpha)$. The error term has been improved a few times; the best error is due to (Zhan, 1993).

The uniformity in α here is perhaps a coincidence – caused by the fact that $1^2 = 1$.

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Conjecture (S., 2021)

Let $k \geqslant 0$ and $\alpha = a/q$ be as above. Then,

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2} + it, \alpha\right)\right|^{2k} dt \sim c_k(\alpha) T (\log T)^{k^2},$$

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Note that $c_k(\alpha)$ does not depend on a!

Some notation

$\boldsymbol{\ell} = \{\ell_\chi\}_{\chi \bmod q}$	Tuples in $\mathbb{Z}_{\geqslant 0}$ indexed by characters mod q
$\lvert\boldsymbol{\ell}\rvert$	$= \sum_\chi \ell_\chi$
$\lambda(\boldsymbol{\ell})$	$= \sum_\chi \ell_\chi^2$
$\mathcal{L}^{\boldsymbol{\ell}}(s)$	$= \prod_\chi L(s,\chi)^{\ell_\chi}$
$d_{\boldsymbol{\ell}}(n)$	Coefficient of n^{-s} in $\mathcal{L}^{\boldsymbol{\ell}}(s)$.

Reduction to mean-square of $\mathcal{L}^\ell(s)$

By orthogonality of Dirichlet characters, we have for $\alpha = a/q$ and $\sigma > 1$,

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$$\begin{aligned}
\zeta(s,\alpha) &= \sum_{n\geqslant 0} \frac{1}{(n+\alpha)^s} = \sum_{n \geqslant 0} \frac{q^s}{(qn+a)^s} \\
&= \sum_{\substack{m \geqslant a \\ m \equiv a \pmod q}} \frac{q^s}{m^s} \\
&= q^s \sum_{m \geqslant 1} \frac{1}{m^s} \left(\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) \chi(m) \right) \\
&= \frac{q^s}{\varphi(q)} \sum_{\chi} \overline{\chi}(a) L(s,\chi).
\end{aligned}$$

By analytic continuation, this holds everywhere in $\mathbb{C}$.

Reduction to mean-square of $\mathcal{L}^\ell(s)$

Thus, by the multinomial theorem,

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where $\binom{k}{\boldsymbol{\ell}}$ are the multinomial coefficients

$$\frac{k!}{\prod_\chi \ell_\chi!}.$$

Reduction to mean-square of $\mathcal{L}^{\boldsymbol{\ell}}(s)$

Using $|\zeta(s,\alpha)|^{2k} = \zeta(s,\alpha)^k \overline{\zeta(s,\alpha)}^k$,

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where $\mathfrak{s}(a; \boldsymbol{\ell}^{(1)}, \boldsymbol{\ell}^{(2)})$ is a sign (complex number of magnitude 1). If we now put $s = \frac{1}{2} + it$ and integrate over $t \in [T, 2T]$, we get $M_k(T;\alpha)$ on the left, while on the right we expect terms with $\boldsymbol{\ell}^{(1)} \neq \boldsymbol{\ell}^{(2)}$ to oscillate very fast.

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where $\mathfrak{s}(a;\boldsymbol{\ell}^{(1)},\boldsymbol{\ell}^{(2)})$ is a sign (complex number of magnitude 1). If we now put $s = \frac{1}{2} + it$ and integrate over $t \in [T, 2T]$, we get $M_k(T;\alpha)$ on the left, while on the right we expect terms with $\boldsymbol{\ell}^{(1)} \neq \boldsymbol{\ell}^{(2)}$ to oscillate very fast. Further, note that $\mathfrak{s}(a;\boldsymbol{\ell},\boldsymbol{\ell}) = 1$.

Reduction to mean-square of $\mathcal{L}^{\ell}(s)$

This gives, heuristically,

$$M_k(T;\alpha) \approx \frac{q^k}{\varphi(q)^{2k}} \sum_{|\boldsymbol{\ell}|=k} \binom{k}{\boldsymbol{\ell}}^2 \int_T^{2T} \left|\mathcal{L}^{\boldsymbol{\ell}}\left(\tfrac{1}{2}+it\right)\right|^2 dt.$$

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Note that the right hand side does not depend on a, as predicted in our conjecture!

Previous results on products of L-functions

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Since Dirichlet L-functions fall in both these classes, their results apply also to $\mathcal{L}^{\ell}(s)$ (and, in fact, also to the Dedekind zeta functions $\zeta_K(s)$ of a Galois number field K).

The main theorem

Theorem (S., 2021)

Under some reasonable conjectures[a] *we have that for any ℓ,*

$$\frac{1}{T}\int_T^{2T} |\mathcal{L}^{\ell}(\tfrac{1}{2}+it)|^2\,dt \sim_{q,k} c_{\ell}(q)\left\{\prod_{\chi}(\log q^*(\chi)T)^{\ell_\chi^2}\right\},$$

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[a] To be described; based on the approach of (Gonek–Hughes–Keating, 2007) instead of the CFKRS recipe.

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Thus, our main conjecture follows from this theorem after some book-keeping.

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$$\mathcal{L}^\ell(s) \approx \mathcal{P}_X^\ell(s)\mathcal{Z}_X^\ell(s),$$

where

$\mathcal{P}_X^\ell(s)$	approximate Euler product	Primes	$p \leqslant X$
$\mathcal{Z}_X^\ell(s)$	approximate Hadamard product	Zeroes	$\lvert\rho - t\rvert \leqslant \frac{1}{\log X}$

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Conjecture (Splitting)

Let $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for any tuple of nonnegative integers ℓ indexed by characters modulo q, we have for $s = 1/2 + it$,

$$\frac{1}{T}\int_T^{2T} \left|\mathcal{L}^{\ell}(s)\right|^2 dt \sim \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{P}_X^{\ell}(s)\right|^2 dt\right) \times \left(\frac{1}{T}\int_T^{2T} \left|\mathcal{Z}_X^{\ell}(s)\right|^2 dt\right).$$

Mean-square of $\mathcal{P}_X^\ell(s)$

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For integer $\ell_\chi \geqslant 0$ such that $|\boldsymbol{\ell}| = \sum_\chi \ell_\chi = k$, further, suppose that $2 \leqslant X \ll_\epsilon (\log T)^{2-\epsilon}$.

$$\frac{1}{T}\int_T^{2T} |\mathcal{P}_X^{\boldsymbol{\ell}}(\tfrac{1}{2}+it)|^2\,dt = b(\boldsymbol{\ell})F_X(\boldsymbol{\ell})\left(1+\mathcal{O}_{q,k,\epsilon}\left(\frac{1}{\log X}\right)\right)$$

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where $b(\boldsymbol{\ell})$ *is an explicit Euler product independent of* X*, and*

$$F_X(\boldsymbol{\ell}) = (e^\gamma \log X)^\lambda \prod_p \left(1 - \frac{1}{p}\right)^{\lambda - |d_{\boldsymbol{\ell}}(p)|^2}.$$

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Here γ is the Euler-Mascheroni constant, $d_{\boldsymbol{\ell}}(n)$ is the coefficient of n^{-s} in the Dirichlet series for $\mathcal{L}^{\boldsymbol{\ell}}(s)$, and $\lambda = \sum_\chi \ell_\chi^2$.

Mean-square of $\mathcal{Z}_X^{\boldsymbol{\ell}}(s)$

The random matrix theory analogy gives us the following conjecture

Conjecture

Suppose that $X, T \to \infty$ with $X \ll_\epsilon (\log T)^{2-\epsilon}$. Then, for $\boldsymbol{\ell}$ as before,

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Other evidence for the conjecture

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2. We prove some small ($|\boldsymbol{\ell}| \leqslant 2$) cases of the splitting and random matrix theory conjectures using standard techniques.
3. We verify that our conjectural constants match up in all the cases where asymptotics are known.

Pseudomoments of $\zeta(s, \alpha)$, $\alpha \notin \mathbb{Q}$

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We thus need to understand a fairly complicated problem of a logarithmic weight count of integer points in a variety. It is still unclear how these weighted counts behave for large k, but something can be said if we drop the logarithmic weights $1/(n_j+\alpha)$.

The unweighted integer-point counting problem

Theorem (Heap–S.–Wooley, 2021)

Let $k \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $\alpha \in \mathbb{C}$ is algebraic of degree d over $\mathbb{Q}$ where $k > d$.

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Here $T_k(X) = k!X^k + O_k(X^{k-1})$ is the number of pairs $(\boldsymbol{n}, \boldsymbol{m})$ with $1 \leqslant n_j, m_j \leqslant X$, $1 \leqslant j \leqslant k$ and $\boldsymbol{n}$ is a permutation of $\boldsymbol{m}$.

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This leads us to the following conjecture.

Conjecture (Heap–S., 2022+)

Let $k \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ be an irrational number. Then for algebraic α of degree $d \geqslant k$ and almost all transcendental α we have

$$M_k(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^{2k} dt \sim k!\, T(\log T)^k$$

as $T \to \infty$.

As mentioned earlier, note that $1^2 = 1$, and so the main term of $M_1(T;\alpha)$ is uniform in $0 < \alpha \leqslant 1$.

The fourth moment of $\zeta(s, \alpha)$ for irrational α

In ongoing work, we have the following theorem.

Theorem (Heap–S, 2022+)

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Let $0 < \alpha < 1$ *be an irrational number.*

The fourth moment of $\zeta(s, \alpha)$ for irrational α

In ongoing work, we have the following theorem.

Theorem (Heap–S, 2022+)

Let $0 < \alpha < 1$ *be an irrational number. Then, under certain Diophantine conditions*[a]*, we have that*

$$M_2(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^4 dt$$

The fourth moment of $\zeta(s, \alpha)$ for irrational α

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$$M_2(T;\alpha) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it, \alpha)|^4 dt \sim 2T(\log T)^2,$$

as $T \to \infty$.

The fourth moment of $\zeta(s,\alpha)$ for irrational α

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Let $0 < \alpha < 1$ be an irrational number. Then, under certain Diophantine conditions[a]*, we have that*

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[a] WIP

The fourth moment of $\zeta(s, \alpha)$ for irrational α

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[a] WIP

In particular, our Diophantine conditions appear to be satisfied by almost all α, so this verifies the previous conjecture for $k = 2$.

Thank You!