ONE HALF LOG DISCRIMINANT

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Abstract. We give a geometric proof that one may compute a particular generalized Mahler integral using equidistribution of preperiodic points of a dynamical system on the sphere. The dynamical system is associated to the multiplication by 2 map on an elliptic curve over a number field K with Weierstrass equation $y^2 = P(x)$ (a Lattès dynamical system). At each finite place v, we prove the local equidistribution formula

$$\frac{1}{2}v(\Delta) = \lim_{n\to\infty} \frac{1}{n^2}(D.H_n)_v,$$

where H_n is the Zariski closure in $\mathbb{P}^1_{\mathcal{O}_K}$ of the image in $\mathbb{P}^1_K$ of the n-torsion minus the 2-torsion and Δ is the discriminant of the polynomial $P(x)$. One consequence of this result is the formula

$$\frac{1}{2}\log|\operatorname{Norm}_{K/\mathbb{Q}}(\triangle)| = \sum_{\sigma:K\to\mathbb{C}} \int_{\mathbb{P}^1(\mathbb{C})} \log|P(x)|_\sigma \frac{dx\wedge d\bar{x}}{\Im(\tau)_\sigma |P(x)|^2_\sigma}.$$

In [21], Szpiro, Ullmo, and Zhang proved that for any abelian variety A over $\mathbb{Q}$, any continuous function g on $A(\mathbb{C})$, and any nonrepeating sequence of point $\beta_n \in A(\overline{\mathbb{Q}})$ with Néron-Tate height tending to zero, one has

$$\frac{1}{|\operatorname{Gal}(\beta_n)|}\sum_{\sigma\in\operatorname{Gal}(\beta_n)} g(\sigma(\beta_n)) = \int_{A(\mathbb{C})} g\, d\mu,$$

where $d\mu$ is the normalized Haar measure on A and $\operatorname{Gal}(\beta_n)$ is the Galois group of the Galois closure of $\mathbb{Q}(\beta_n)$ in $\mathbb{C}$. This result says, in effect, that Galois orbits of points with small Néron-Tate height are equidistributed in $A(\mathbb{C})$. Ullmo [22] and Zhang [23] later used this fact to give proofs of the Bogomolov conjecture for abelian varieties.

When the abelian variety A is an elliptic curve, the multiplication by 2 map gives rise to a map on the projective line, called a Lattès map. Thus, in this case, the work of [21] can be viewed as an equidistribution result for a rational map on the projective line. Recently, a variety of authors have proven more general equidistribution results for

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arbitrary rational maps of degree greater than 1 on the projective line; see Autissier [2], Baker/Rumely [5], Bilu [6], Chambert-Loir [7], and Favre/Rivera-Letelier [12, 11], for example. Many of these results hold for measures at finite places as well as at archimedean places.

In [15], it is shown that for any nonconstant map $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree greater than 1 over a number field K, the canonical height $h_\varphi(\alpha)$ of an algebraic point α with minimal polynomial F can be calculated by integrating $\log |F|$ along the invariant measures for the map φ. This gives a generalization of the notion of a Mahler measure of a polynomial (see [14]). Everest, Ward, and Fhlathuin [10, 9] had previously extended the notion of Mahler measures to elliptic curves.

Additional difficulties arise, however, when one attempts to prove equidistribution results for the functions $\log |F|_v$. Indeed, the exact analog of the main result of [21] is not true when the continuous function g is replaced by a function of the form $\log |F|$ (see [1] or [4]). On the other hand, it is possible to prove an equidistribution result for functions of the form $\log |F|_v$ provided that one averages over all points of period n as n goes to infinity rather than over Galois orbits of families of points of small height (see [19]). In the case of elliptic curves, Baker, Ih, and Rumely [4] were able to refine this to prove that for any algebraic number α and any Lattès map φ one has

$$[K(\alpha) : \mathbb{Q}]\, h_\varphi(\alpha) = \sum_{\text{places } v \text{ of } K} \lim_{n\to\infty} \frac{1}{|\operatorname{Gal}(\beta_n)|} \sum_{\sigma \in \operatorname{Gal}(\beta_n)} \log |F(\beta_n^\sigma)|_v$$

for any nonrepeating sequence of algebraic points β_n such that $h_\varphi(\beta_n) = 0$ for all n. Both [4] and [19] use results from diophantine approximation, specifically Roth's theorem ([16]) and A. Baker's work on linear forms in logarithms ([3]).

When one applies the results of [4] and [15] to the points of period 2 for a Lattès map corresponding to multiplication by 2 on the elliptic curve E with Weierstrass equation $y^2 = P(x)$, one obtains the formula

$$\begin{aligned} \frac{1}{2} \log |\operatorname{Norm}_{K/\mathbb{Q}}(\triangle)| &= \sum_{\sigma : K \hookrightarrow \mathbb{C}} \lim_{n\to\infty} \frac{1}{n^2} \log \prod_{\beta \in \operatorname{Supp} H_n} |P(\beta)|_\sigma \\ &= \sum_{\sigma : K \hookrightarrow \mathbb{C}} \int_{\mathbb{P}^1(\mathbb{C})} \log |P(x)|_\sigma \, \frac{dx \wedge d\bar{x}}{\Im(\tau)_\sigma |P(x)|_\sigma^2}, \end{aligned}$$

where Δ is the discriminant of F over K and τ_σ denotes the element corresponding to the elliptic curve E_σ in the fundamental domain for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the Poincaré upper half space in $\mathbb{C}$. Using the product formula and the fact that $h_\varphi(\alpha) = 0$ for periodic points α, this is equivalent to showing that at each nonarchimedean place v of

K, we have

$$\lim_{n\to\infty}\frac{1}{n^2}\log\prod_{\beta\in\operatorname{Supp}H_n}|P(\beta)|_\sigma = \lim_{n\to\infty}\frac{1}{n^2}\log\mathrm{Norm}_{H_n/\mathcal{O}_K}(P_{|H_n})$$
$$= \lim_{n\to\infty}\frac{1}{n^2}\sum_v (D.H_n)_v\log\mathrm{N}(v),$$

where $N(v)$ is the cardinality of the residue field at v and H_n is the Zariski closure in $\mathbb{P}^1_{\mathcal{O}_K}$ of the image in $\mathbb{P}^1_K$ of the n-torsion minus the 2-torsion

We give here a local proof using blow-ups of closed points and intersection theory on $\mathbb{P}^1_V$. This proof uses resolution of singularities (in fact separation of branches) of one-dimensional schemes by blowing up. It also uses information about the special fiber of an elliptic curve with semistable reduction (see [17]). We do not use diophantine approximation. The proof we give is valid for equicharacteristic V (local geometric case) as well as for unequal characteristic (local arithmetic case). Note that the case of positive characteristic does not follow from the results of [4] and [19], since the relevant approximation theorems are not valid in characteristic p. Relations between the discriminant of an elliptic curve and its n-torsion have been studied in [13] and in [20]. The main theorem of this paper is the following.

Theorem. *Let V be a discrete valuation ring with fraction field K. Let $y^2 = P(x)$ be the minimal Weierstrass equation with coefficients in V of an elliptic curve E. Suppose that E has semi-stable reduction over V, that the roots of $P(x)$ are rational over K, and that the residual characteristic of V is not 2. Let D be the scheme of zeroes of $P(x)$ in $\mathbb{P}^1_V$ and let H_n be the Zariski closure in $\mathbb{P}^1_V$ of the image in $\mathbb{P}^1_K$ of the kernel of the multiplication by n in E_K minus the 2-torsion. Then*

$$\lim_{n\to\infty}\frac{1}{n^2}(D.H_n)_v = \frac{1}{2}v(\triangle),$$

where v is the normalized valuation of V, $\triangle$ is the discriminant of E over V, and $(-.-)_v$ is the geometric intersection pairing on the surface $\mathbb{P}^1_V$.

The conditions on the rationality of the roots of P and on the characteristic of V are not essential for the theorem but they make the proof easier. They imply that the valuation of the discriminant is even; we write $2k = v(\triangle)$. One knows (see for example [17] or [8]) that the closed fiber of the minimal model $\mathcal{E}$ for E over V is a cycle of $2k$ projective lines of self intersection (-2); it is obtained by blowing up the plane model for the elliptic curve k times. Recall that the Néron model in this case is

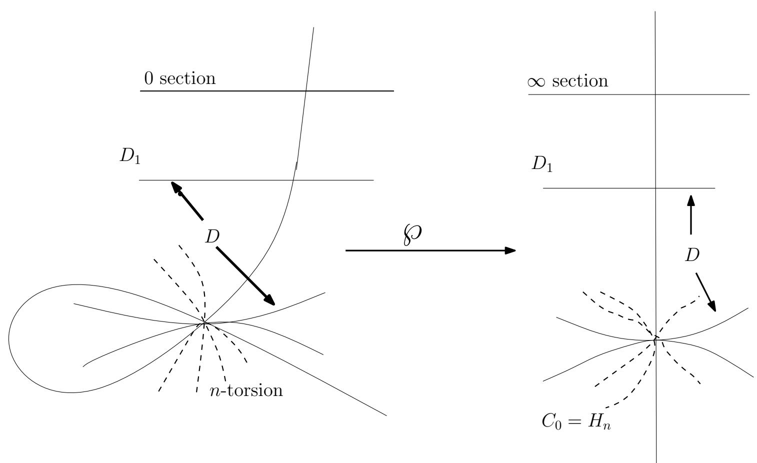
$$\mathcal{E} \setminus \{\text{singular points of the special fiber}\} \text{ (see figure 3).}$$

The strategy of the proof is to compute intersections in the Néron model for E after a suitable base change. The multiplicative structure of the special fiber is simply $\mathbb{G}_m$ crossed with the group of components. One can easily see how the n-torsion distributes itself among the components, and that allows one to calculate intersections without difficulty.

We will use the fact that the Néron model for E has $2k$ components in its special fiber(see [17, page 365]). It is naturally a $2:1$ cover of a model for $\mathbb{P}^1$ with $k+1$ components. The hyperelliptic map induces a map on components that sends a component and its inverse to a single component; there are two components that are their own inverse (the identity and the component of order 2), which gives a total of $2(k-1)/2+2 = k+1$ components on a model of $\mathbb{P}^1$.

We begin with the plane model E for E_K coming from the equation $y^2 = P(x)$. In this case, the special fiber is simply a nodal cubic because we have not begun blowing up yet. Here, and elsewhere in the figures, we depict sections (such as D_1 and the 0 and ∞ sections) as horizontal lines, each of which meets the special fiber in a point.

Figure 1



E – a plane model of the degenrate fibre

x-axis

Definition 1. *Let D_0 denote the divisor D. We define the divisor D_i recursively (for $i \leq k$) as the proper transform of D_{i-1} for the blow-up $\sigma_i : X_i \to X_{i-1}$ centered at the point P_{i-1} of multiplicity 2 on D_{i-1}.*

Note that this is a horizontal divisor of degree 3 which intersects the special fiber F_0 of $\mathbb{P}^1_V = X_0$ in 2 points: one P_0 of multiplicity 2 on D_0, the other one of multiplicity 1 on D_0. We now define the divisors in our models X_i coming from H_n.

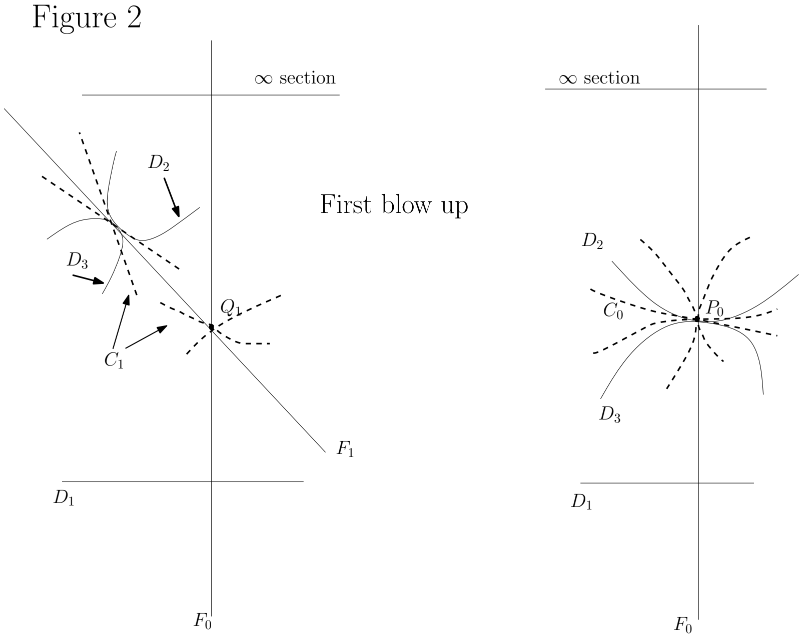
Definition 2. *The horizontal divisor C_0 is defined to be H_n for some fixed odd n. The divisor C_i is the proper transform of C_{i-1} in X_i.*

The degree of C_0 (on the generic fiber) is $(n^2 - 1)/2$ when n is odd and $(n^2/2) - 3$ when n is even. This follows from the fact that the hyperelliptic map sends each point and its inverse to the same point in $\mathbb{P}^1$.

Definition 3. *Define $\wp_K : E_K \to \mathbb{P}^1_K$ to be the projection onto "the x axis" (i.e., $\wp$ is the Weierstrass $\wp$ function). We will, by abuse of language, note $\wp : E_i \to X_i$ to be the the extension of $\wp_K$ to model E_i for E_K over V.*

Figure 2 below illustrates what happens when one first begins to blow-up the model for $\mathbb{P}^1$. Here F_0 denotes the original special fiber on right and its strict tranform on the left, while F_1 is the exceptional divisor.

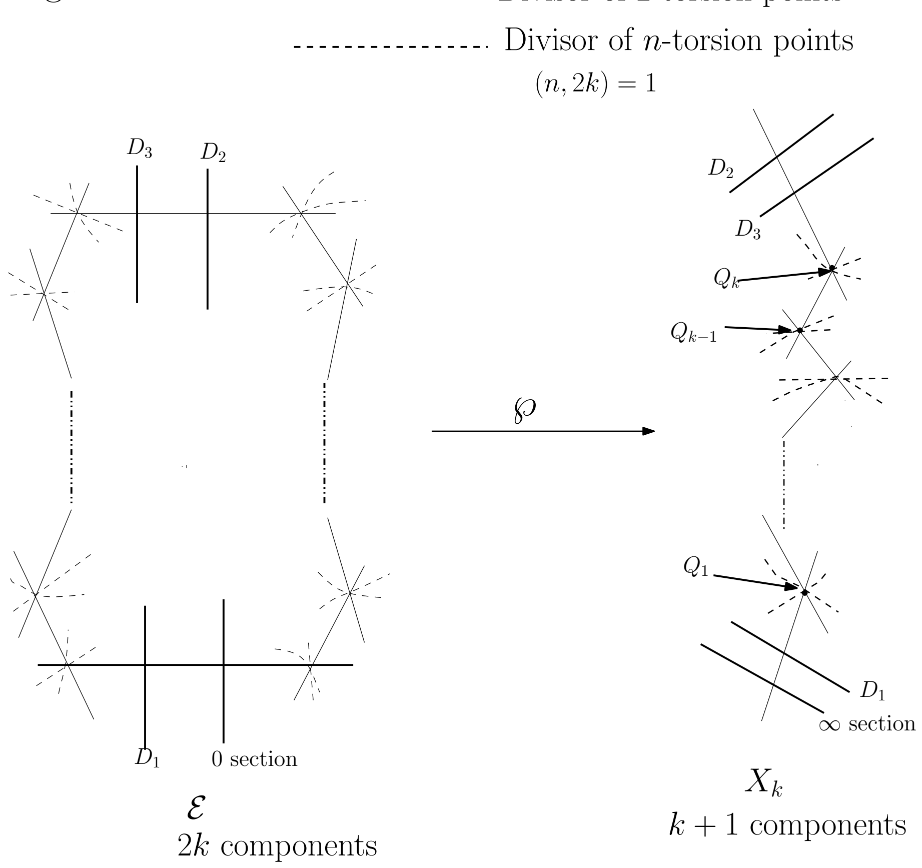
Figure 2



Lemma. *Assume that n is odd or that the residual characteristic is not 2, then after k successive blow-ups of the points P_i of multiplicity 2 on D_i, the proper transform D_k is étale and the proper transforms D_k and C_k do not meet.*

Proof. (*Of Lemma.*) If $\wp^*(C_k)$ and $\wp^*(D_k)$ are both in the Néron model (i.e., if n and $(2k)$ have a common factor m), then H_n and 2-torsion are distinct; hence, when the characteristic is not 2, they do not meet in the Néron model. If n is prime to $2k$ and $\wp^*(H_k)$ is not inside the Néron model, then $\wp^*(H_k) \bigcap \wp^*(D_k) = \emptyset$, since $\wp^*(D_k)$ is in the Néron model (see figure 3 below and recall our earlier explanation about the number of components in the special fibers of the model $\mathcal{E}$ for E and the model X_K for the projective line). $\square$

Figure 3



We are now ready to prove the main theorem.

Proof. We treat first the case when n is prime to $2k$. The exceptional divisor of σ_i will be denoted as F_i. By abuse of language the proper transform of F_i will still be called F_i after $\sigma_{i+1}, \dots, \sigma_k$. We will let Q_i denote the point of intersection of F_i with F_{i-1} in X_i (see figure 2).

We will denote the usual pull-back map for divisors with $*$. We denote the composed map $\sigma_i \cdot \sigma_{i-1} \cdots \sigma_1$ as ρ_i. After i blow-ups, one has integers $m_{j,i}$ such that

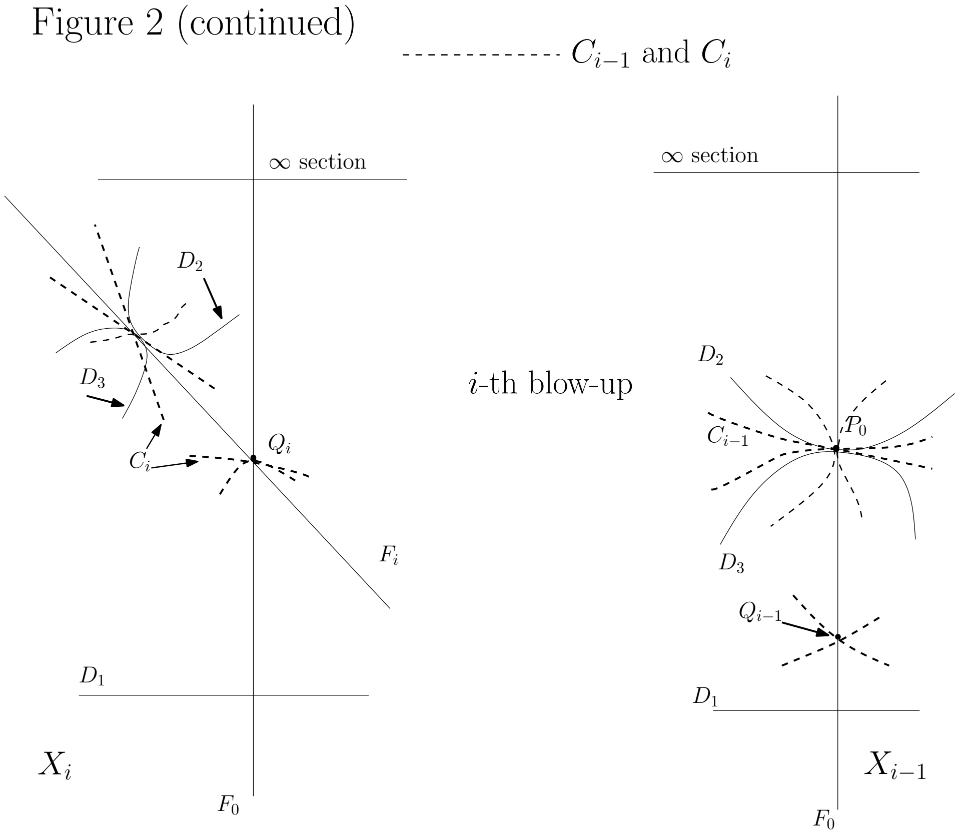
$$\rho_i^* D = D_i + \sum_{j \leq i} m_{j,i} F_j$$

and

$$\pi_{i+1}^* D = \sigma_{i+1}^* D_i + \sum_{j \leq i} m_{j,i} \sigma_{i+1}^* F_j .$$

The figure below illustrates what happens when we take the i-th blow-up. This is quite similar to what happens after the first blow-up: one obtains an additional component in the special fiber coming from the exceptional divisor.

Figure 2 (continued)



As long as i is less than $(k-1)$, one has

$$\sigma_{i+1}^* D_i = D_{i+1} + 2F_{i+1},$$

since the multiplicity of D_i at P_i is still 2. Since $\sigma_{i+1}^* F_i = F_i + F_{i+1}$ we have $m_{j,i} = m_{j,j}$ for any $i \geq j$, so we have

$$m_{j,i} = m_{j,j} = m_{j-1,j-1} + 2$$

for all $i \geq j$. Thus, by induction, we have $m_{j,j} = 2j$ for each j, which means that $m_{j,i} = 2j$ for all $i \geq j$.

The intersection multiplicity we are looking for can be computed as follows

$$(*) (H_n.D) = (C_0.D_0) = (C_k.(D_k + \sum_{j \leq k} m_{j,k} F_j)) = 2 \sum_{j \leq k} j(C_k.F_j).$$

One is left with computing each $(C_k.F_j)$. We will achieve this by looking at the special fiber of various models of E_K over V. By the projection formula for $\wp$ we can compute intersections on the minimal model $\mathcal{E}$ of E_K or on the k-th blow-up X_k of $\mathbb{P}^1$. In fact we will use the projection formula to compute intersections on the minimal model $\mathcal{E}'$ for E after the base change $\operatorname{Spec} V[X]/(X^n - \pi) \to \operatorname{Spec} V$ where π is a uniformizing parameter of V. A description of the resolution of singularities of the base change can be found in [18, Exposé 1, Proposition 2.2].

On the minimal model $\mathcal{E}'$, the special fiber has $2kn$ components. Let Z_0 denote the component of the origin of the elliptic curve, and let us denote the other components as $Z_1, \ldots, Z_{2kn-1}$ in such a way that Z_i meets Z_{i+1} for $0 \leq i \leq (2kn-1)$ and Z_{2k-1} meets Z_0 (figure 4).

The divisor of n-torsion points meets only the components Z_i for which i is a multiple of $2k$; the multiplicity of each intersection is n. The components Z_j for which j is a multiple of n are the only ones not contracted by the morphism to the plane model E. The contribution at Q_j in the intersection number $(C_k.F_j)$ for $j \neq 0, k$ will be

$$n \cdot \left| \{ m \text{ such that } (j-1)n \leq 2km \leq jn \} \right|$$

Write $n = 2kq + r$ with $0 \leq r < 2k$. We have

$$\left| \left| \{ m \text{ such that } (j-1)n \leq 2km \leq jn \} \right| - q \right| \leq 1.$$

Thus, we have

$$(**) \left| (C_k.F_j) - 2n \frac{n-r}{2k} \right| \leq 2n.$$

Since $(C_k.F_k) = n\frac{n-r}{k}$ we obtain

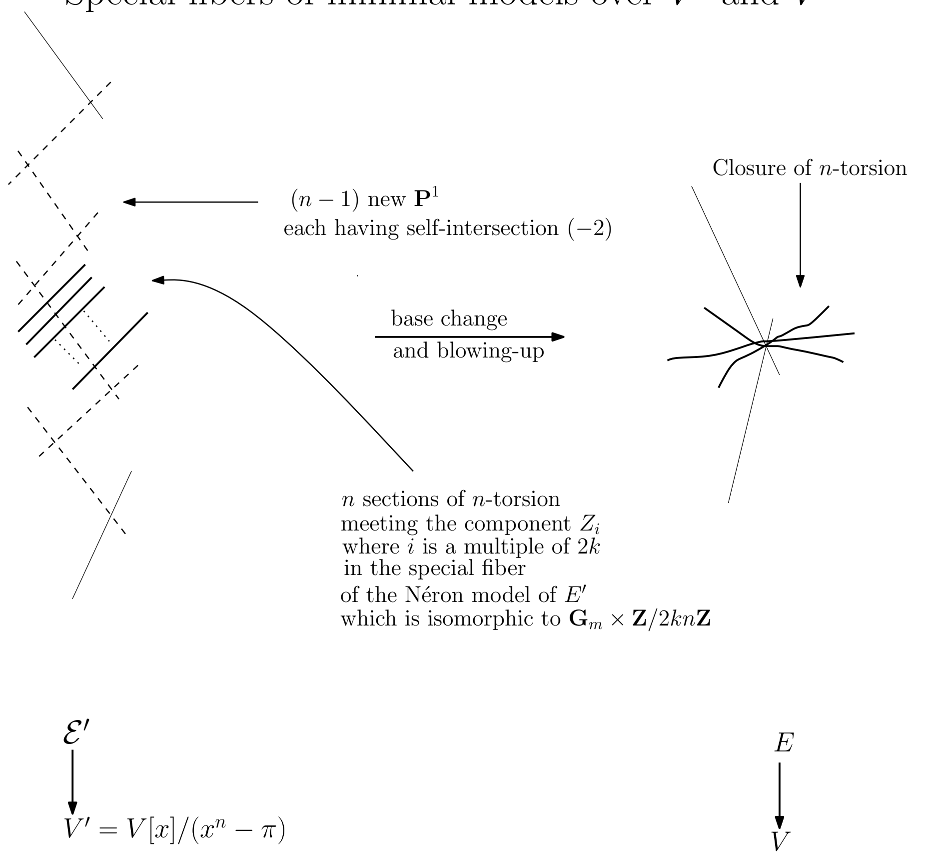
$$\begin{aligned}\left|(H_n.D) - \Big(2\sum_{j\le(k-1)} j2n\frac{n-r}{2k} + 2kn\frac{n-r}{k}\Big)\right| &\le 2\sum_{j\le(k-1)} j(2n)\\ &= 2\frac{k(k-1)}{2}2n.\end{aligned}$$

Hence, we have $|(H_n.D) - (n-r)nk| \le k(k-1)2n$, so

$$\lim_{n\to\infty}\frac{1}{n^2}(H_n.D) = k.$$

Figure 4 below illustrates what happens to the component group when one extends the base to $V' = V[x]/(x^n - \pi)$ from V; simply put, the number of components increases in a way that makes it easier to calculate the relevant intersections.

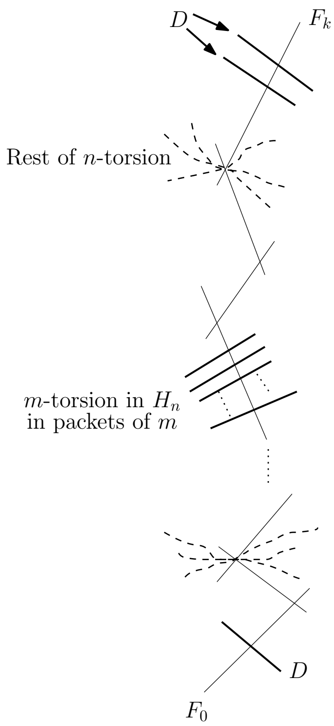
Figure 4
Special fibers of minimal models over V' and V



This finishes the proof in the case when n and $2k$ are relatively prime. For the case where n and $2k$ have a gcd m greater than 1 the formula (*) is still valid. The n-torsion distribute themselves in packets of m in components of the special fiber (as illustrated in figure 5 below). Thus, the estimate (**) for $(C_k.F_i)$ has now an error term of at most m.

Figure 5

$(2k, n) = m$



Adding as before, we now obtain

$$|(H_n.D) - (n-r)nk| \leq \sum_{j \leq (k-1)} 2jm = mk(k-1) \leq k(k-1)(2k).$$

Letting n go to ∞ we see again that

$$\lim_{n\to\infty} \frac{1}{n^2}(H_n.D) = k.$$

$\square$

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