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Preperiodic points for families of rational maps

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Abstract

Let X be a smooth curve defined over $\bar{\mathbb{Q}}$, let $a, b \in \mathbb{P}^1(\bar{\mathbb{Q}})$ and let $f_\lambda(x) \in \bar{\mathbb{Q}}(x)$ be an algebraic family of rational maps indexed by all $\lambda \in X(\mathbb{C})$. We study whether there exist infinitely many $\lambda \in X(\mathbb{C})$ such that both a and b are preperiodic for f_λ. In particular, we show that if $P, Q \in \bar{\mathbb{Q}}[x]$ such that $\deg(P) \geqslant 2 + \deg(Q)$, and if $a, b \in \bar{\mathbb{Q}}$ such that a is periodic for $P(x)/Q(x)$, but b is not preperiodic for $P(x)/Q(x)$, then there exist at most finitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for $P(x)/Q(x) + \lambda$. We also prove a similar result for certain two-dimensional families of endomorphisms of $\mathbb{P}^2$. As a by-product of our method, we extend a recent result of Ingram ['Variation of the canonical height for a family of polynomials', *J. reine. angew. Math.* 685 (2013), 73–97] for the variation of the canonical height in a family of polynomials to a similar result for families of rational maps.

1. *Introduction*

In [**1**], Baker and DeMarco study the following question: given complex numbers a and b, and an integer $d \geqslant 2$, when do there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for the action of $f_\lambda(x) := x^d + \lambda$ on $\mathbb{C}$? They show that this happens if and only if $a^d = b^d$. The problem, originally suggested by Zannier, is a dynamical analog of a question on families of elliptic curves studied by Masser and Zannier in [**20**–**22**]. This problem was motivated by the Pink–Zilber conjectures in arithmetic geometry regarding unlikely intersections between a subvariety V of a semiabelian variety A and families of algebraic subgroups of A of codimension greater than the dimension of V (see [**6**, **16**, **23**]). A thorough treatment of the problem of unlikely intersections on families of semiabelian varieties can be found in [**30**].

The authors extended the results of [**1**] to more general families of polynomials in [**13**]. The polynomials considered in [**1**, **13**] are algebraic families parameterized by points in an affine subset of the projective line. In this paper, we extend our investigation to families of rational maps and the parameter spaces are general algebraic curves defined over a number field. Moreover, general families of two-dimensional endomorphisms of $\mathbb{P}^2$ are also studied. As in [**1**, **13**], a key ingredient in the study of families of mappings is the application of equidistribution theorems [**3**, **9**, **12**, **28**] to the situation of arithmetic dynamics. Note that the equidistribution results of [**3**, **9**, **12**] apply only in dimension 1. As we also treat the case of higher-dimensional parameter spaces in this paper, we apply the equidistribution results, obtained by Yuan [**28**] and the recent result of Yuan–Zhang [**29**] to these more general families of maps. We prove the following higher genus generalization of [**1**, Theorem 1.1].

Theorem 1.1. *Let C be a projective nonsingular curve defined over $\bar{\mathbb{Q}}$, let $\eta \in C(\bar{\mathbb{Q}})$, and let A be the ring of functions on C regular on $C \setminus \{\eta\}$. Let $\Phi, \Psi \in A$ be nonconstant functions.*

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Let $P_i, Q_i \in \bar{\mathbb{Q}}[x]$ *for* $i = 1, 2$ *be polynomials such that* $\deg(P_i) \geqslant \deg(Q_i) + 2$ *for each* i*. Let*

$$\mathbf{f}_\lambda(x) := \frac{P_1(x)}{Q_1(x)} + \Phi(\lambda) \quad \text{and} \quad g_\lambda(x) := \frac{P_2(x)}{Q_2(x)} + \Psi(\lambda)$$

be one-parameter families of rational maps indexed by all $\lambda \in C(\mathbb{C})$*, and let* $a, b \in \bar{\mathbb{Q}}$ *such that both* $Q_1(a)$ *and* $Q_2(b)$ *are nonzero. If there exist infinitely many* $\lambda \in C(\mathbb{C})$ *such that* a *is preperiodic under the action of* $\mathbf{f}_\lambda$ *and* b *is preperiodic under the action of* $\mathbf{g}_\lambda$*, then, for* each $\lambda \in C(\mathbb{C})$, a *is preperiodic under the action of* $\mathbf{f}_\lambda$ *if and only if* b *is preperiodic under the action of* $\mathbf{g}_\lambda$*.*

The following is an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *Let* $a, b \in \bar{\mathbb{Q}}$ *and let* $P_i, Q_i \in \bar{\mathbb{Q}}[x]$ *for* $i = 1, 2$ *be polynomials such that* $\deg(P_i) \geqslant \deg(Q_i) + 2$ *for each* i*. Assume that* a *is preperiodic for* $P_1(x)/Q_1(x)$ *but* $Q_1(a) \neq 0$*, and that* b *is not preperiodic for* $P_2(x)/Q_2(x)$*. Then there exist at most finitely many* $\lambda \in \mathbb{C}$ *such that both* a *is preperiodic for* $P_1(x)/Q_1(x) + \lambda$ *and also* b *is preperiodic for* $P_2(x)/Q_2(x) + \lambda$*.*

Theorem 1.1 will follow from our main result (see Theorem 2.1). Our main result (see Theorem 2.1) also has applications to the study of *post-critically finite* (PCF) maps. A map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is called PCF if each critical point of f is preperiodic under the action of f. The PCF maps are important in algebraic dynamics; recently Baker and DeMarco [**2**] have made a far-reaching conjecture about them. Baker and DeMarco are interested in locating the PCF polynomials within the moduli space $\mathcal{P}_d$ of all polynomial maps of degree d. Dujardin and Favre [**11**] showed that the PCF maps are equidistributed with respect to the bifurcation measure in $\mathcal{P}_d$; in particular, they form a Zariski-dense subset of $\mathcal{P}_d$. Baker and DeMarco aim at characterizing curves (or subvarieties) in $\mathcal{P}_d$ containing a Zariski-dense subset of PCF maps; the expectation is that such subvarieties are very special. Roughly speaking, [**2**] conjectures that a subvariety $V \subset \mathcal{P}_d$ contains a Zariski-dense subset of PCF maps if and only if V is cut out by critical orbit relations. The notion of 'critical orbit relation' is a bit delicate, as one needs to take into account the presence of symmetries in any given family of polynomials. We can prove the following result which offers support to the main conjecture of [**2**] (see also [**14**]).

THEOREM 1.3. *Let* $f, g \in \bar{\mathbb{Q}}[z]$ *be polynomials of degree larger than* 1*, let* $C \subset \mathbb{A}^2$ *be a curve with the property that its projective closure in* $\mathbb{P}^2$ *is a nonsingular curve with exactly one point at infinity. If there exist infinitely many points* $(x, y) \in C(\bar{\mathbb{Q}})$ *such that* $f(z) + x$ *and* $g(z) + y$ *are both PCF maps, then, for each point* $(x, y) \in C(\mathbb{C})$*, we have that* $f(z) + x$ *is PCF if and only if* $g(z) + y$ *is PCF.*

It is not clear, in general, how many of the results above should carry over into higher-dimensional situations. As another application of the techniques developed in this paper, we are able to prove a first result regarding *unlikely intersections* for algebraic dynamics in higher dimensions.

THEOREM 1.4. *Let* $P(X, Z) \in \bar{\mathbb{Q}}[X, Z]$ *and* $Q(Y, Z) \in \bar{\mathbb{Q}}[Y, Z]$ *be homogeneous polynomials of degree* $d \geqslant 3$ *and assume that* $P(X, 0)$ *and* $Q(Y, 0)$ *are nonzero. Let* $\mathbf{f}_{\lambda,\mu} : \mathbb{P}^2 \to \mathbb{P}^2$ *be the two-parameter family defined by*

$$\mathbf{f}_{\lambda,\mu}([X : Y : Z]) = [P(X, Z) + \lambda Y Z^{d-1} : Q(Y, Z) + \mu X Z^{d-1} : Z^d].$$

Let $a_i, b_i \in \bar{\mathbb{Q}}^*$ *(for* $i = 1, 2$*). If there exists a set of points* $[\lambda : \mu : 1]$ *that is Zariski-dense in* $\mathbb{P}^2$ *such that, for each of such pairs* (λ, μ) *both* $[a_1 : b_1 : 1]$ *and* $[a_2 : b_2 : 1]$ *are preperiodic for* $\mathbf{f}_{\lambda,\mu}$*, then, for each* $\lambda, \mu \in \bar{\mathbb{Q}}$, $[a_1 : b_1 : 1]$ *is preperiodic for* $\mathbf{f}_{\lambda,\mu}$ *if and only if* $[a_2 : b_2 : 1]$ *is preperiodic for* $\mathbf{f}_{\lambda,\mu}$.

We sketch briefly the ideas for proving Theorem 2.1. Let $\{\mathbf{f}_\lambda\}$ be an algebraic family of rational maps on the projective line $\mathbb{P}^1$ parameterized by points $\lambda \in Y(\bar{K})$ where Y is an affine subset of the algebraic curve X over a number field K. As mentioned above, we apply recent results of Yuan [**28**] and Yuan–Zhang [**29**] to our situation. The main result of [**28**] shows that points of small height with respect to a semipositive adelic metrized line bundle equidistribute with respect to the measures induced by this semipositive adelic metrized line bundle; on the other hand, [**29**, Theorem 2.10] says that when semipositive metrics on a line bundle induce the same measures at a place, they must differ by a constant. Taken together, these results say, roughly, that if two appropriate height functions on a variety X that come from semipositive adelic metrized line bundles share a Zariski-dense family of points of small heights, then the two height functions must be exactly the same. For a given family of points $\{\mathbf{c}_\lambda\}$, we consider the canonical heights $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_\lambda)$ of $\mathbf{c}_\lambda$ associated to the map $\mathbf{f}_\lambda$. A key observation is that under appropriate conditions, a suitable multiple (depending on $\mathbf{c}$) of $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_\lambda)$ induces a height function $h_{\mathbf{c}}$ coming from a metrized line bundle on X (see Subsection 5.2 for details). It follows that $\mathbf{c}_\lambda$ is a preperiodic point for $\mathbf{f}_\lambda$ if and only if $h_{\mathbf{c}}(\lambda) = 0$. Now, let $\mathbf{c}_1$ and $\mathbf{c}_2$ be two given families of points. For simplicity, here we only consider one family of rational maps ($\mathbf{f}_1 = \mathbf{f}_2$ in the statement of the theorem) on $\mathbb{P}^1$ and assume that there are infinitely many $\lambda \in Y(\bar{K})$ such that both $\mathbf{c}_{1,\lambda}$ and $\mathbf{c}_{2,\lambda}$ are preperiodic for $\mathbf{f}_\lambda$. Let $h_{\mathbf{c}_i}, i = 1, 2$, be the corresponding heights on X induced from $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_{i,\lambda}), i = 1, 2$, respectively. Then, the infinite set of parameters λ such that both $\mathbf{c}_{1,\lambda}$ and $\mathbf{c}_{2,\lambda}$ are preperiodic points for $\mathbf{f}_\lambda$ yields a Zariski-dense set of small points on X. Using the results of Yuan [**28**] and Yuan–Zhang [**29**] mentioned above, we conclude that the two height functions $h_{\mathbf{c}_1}$ and $h_{\mathbf{c}_2}$ are actually equal. Then we deduce that $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_{1,\lambda}) = 0$ if and only if $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_{2,\lambda}) = 0$ which concludes the proof of Theorem 2.1. Thus, our strategy follows that of [**1**], but in the language of adelic metrized line bundles rather than Green functions.

As a consequence of our method, using the notation from the previous paragraph, we prove

$$\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = \hat{h}_{\mathbf{f}}(\mathbf{c}) \cdot h_{\mathbf{c}}(\lambda) + O(1), \tag{1.1}$$

where $\hat{h}_{\mathbf{f}}(\mathbf{c})$ is the canonical height of $\mathbf{c} \in \mathbb{P}^1(F)$ under the action of $\mathbf{f} : \mathbb{P}^1 \to \mathbb{P}^1$ (where $F = K(X)$). Formula (1.1) (which is proved in Theorem 5.4) extends a recent result of Ingram [**18**] to certain families of rational maps. See Subsection 5.2 for the statement of Theorem 5.4 and a discussion of Ingram's result [**18**] (which in turn extends previous results of Silverman [**24**] and Call–Silverman [**7**]).

The plan of our paper is as follows. In Section 2, we state Theorem 2.1 (which generalizes Theorem 1.1) and also state a few of its consequences. We also make a conjecture that would generalize the results of this paper and place it in a more natural context. Section 3 describes the notation that is used throughout the paper. Then, in Section 4 we introduce metrized line bundles and state Yuan's [**28**] equidistribution result and Yuan–Zhang's [**29**] 'Calabi–Yau' result on metrics that give rise to the same measure. Section 5 is devoted to setting up our problem so that the results from the previous section can be applied. In Sections 6 and 7, we prove that our metrics satisfy the necessary hypotheses that allow us to use the results of [**28**, **29**]. Section 8 contains a proof of Theorem 2.1 and of its consequences. In Section 9, we prove Theorem 1.4.

2. *Statement of the main results*

We start with the statement of our main result. Recall that a point $P \in \mathbb{P}^1$ is a superattracting periodic point for $\mathbf{f}$ if P is periodic of period n for $\mathbf{f}$ and that $(\mathbf{f}^n)'(P) = 0$. Then, we have the following.

THEOREM 2.1. *Let K be a number field and let X be a smooth projective curve defined over K. Let $\eta \in X(K)$ and let $Y := X \setminus \{\eta\}$. We let $\mathbf{A}$ be the ring of rational functions on X that are regular on Y, defined over K. For each $\mathbf{a} \in \mathbf{A}$, we denote by $\deg(\mathbf{a}) = -\operatorname{ord}_\eta(\mathbf{a})$ where ord_η is the order of the pole η for the function $\mathbf{a}$.*

Suppose that we have rational functions $\mathbf{f}_1 = P_1(x)/Q_1(x)$ and $\mathbf{f}_2 = P_2(x)/Q_2(x)$ such that $P_i, Q_i \in \mathbf{A}[x]$ and the leading coefficients of P_i and of Q_i are nonzero constants for $i = 1, 2$. Furthermore, assume that $\mathbf{f}_1$ and $\mathbf{f}_2$ satisfy the following conditions.

(1) *The resultant $R(\mathbf{f}_i) := \operatorname{Res}(P_i(x), Q_i(x); x)$ of $P_i(x)$ and $Q_i(x)$ is a nonzero constant (that is, $R(\mathbf{f}_i) \in K^*$).*
(2) *The point $x = \infty$ is a superattracting fixed point for both $\mathbf{f}_1$ and $\mathbf{f}_2$.*

For each $\lambda \in Y(\bar{K})$ and $i = 1, 2$, we denote by $\mathbf{f}_{\lambda,i}$ the rational function obtained by evaluating each coefficient of P_i and of Q_i at λ. We denote by $\hat{h}_{\mathbf{f}_{\lambda,i}}$ the canonical height associated to the function $\mathbf{f}_{\lambda,i}$.

Let $\mathbf{c}_i = \mathbf{a}_i/\mathbf{b}_i$ where $\mathbf{a}_i, \mathbf{b}_i \in \mathbf{A}$ and

$$(\mathbf{a}_i, \mathbf{b}_i) = \mathbf{A} \quad \text{for } i = 1, 2, \tag{2.1}$$

and suppose that the two sequences $\{\deg(\mathbf{f}_i^n(\mathbf{c}_i)) \mid n \in \mathbb{N}\}$ are not bounded. If there exists an infinite family of $\lambda_n \in Y(\bar{K})$ such that

$$\lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n,1}}(\mathbf{c}_1(\lambda_n)) = \lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n,2}}(\mathbf{c}_2(\lambda_n)) = 0, \tag{2.2}$$

then, for all $\lambda \in Y(\bar{K})$, we have $\hat{h}_{\mathbf{f}_{\lambda,1}}(\mathbf{c}_1(\lambda)) = 0$ if and only if $\hat{h}_{\mathbf{f}_{\lambda,2}}(\mathbf{c}_2(\lambda)) = 0$.

REMARKS 2.2. (1) Condition (2) in Theorem 2.1 is equivalent to $\deg_x P_i(x) \geqslant \deg_x Q_i(x) + 2$ for both $i = 1, 2$.

(2) Theorem 2.1 yields that under the above hypotheses, $\mathbf{c}_1(\lambda)$ is preperiodic under $\mathbf{f}_{\lambda,1}$ if and only if $\mathbf{c}_2(\lambda)$ is preperiodic under $\mathbf{f}_{\lambda,2}$ since a point has canonical height equal to 0 if and only it is preperiodic.

We believe that Theorem 2.1 should hold in a more general situation. Namely, as long as neither $\mathbf{c}_1$ nor $\mathbf{c}_2$ is a (persistent) preperiodic point for $\mathbf{f}_1$, respectively, for $\mathbf{f}_2$ and there is an infinite sequence of λ_n satisfying (2.2), then the conclusion of Theorem 2.1 should hold. Furthermore, if there is only one family of morphisms (that is, $\mathbf{f}_1 = \mathbf{f}_2$), then (2.2) should yield more precise relation between $\mathbf{c}_1$ and $\mathbf{c}_2$. For this, we make the following conjecture.

CONJECTURE 2.3. Let Y be any quasiprojective curve defined over $\bar{\mathbb{Q}}$, and let F be the function field of Y. Let $\mathbf{a}, \mathbf{b} \in \mathbb{P}^1(F)$ and let $V \subset \mathcal{X} := \mathbb{P}^1_F \times_F \mathbb{P}^1_F$ be the $\bar{\mathbb{Q}}$-curve $(\mathbf{a}, \mathbf{b})$. Let $\mathbf{f} : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geqslant 2$ defined over F. If there exists an infinite sequence of points $\lambda_n \in Y(\bar{\mathbb{Q}})$ such that $\lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n}}(\mathbf{a}(\lambda_n)) = \lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n}}(\mathbf{b}(\lambda_n)) = 0$, then V is contained in a proper preperiodic subvariety of $\mathcal{X}$ under the action of $\Phi := (\mathbf{f}, \mathbf{f})$.

To support our Conjecture 2.3, we prove Theorem 2.4 generalizing the main results in [**1**, **13**] in the case where $\mathbf{f}$ is a polynomial map on $\mathbb{P}^1$. Note that in the conjecture above, $\mathbf{f}$ induces a well-defined rational map $\mathbf{f}_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ defined over $\bar{\mathbb{Q}}$ for all but finitely many $\lambda \in Y(\bar{\mathbb{Q}})$; as usual, $\hat{h}_{\mathbf{f}_\lambda}$ is the (global) canonical height corresponding to the rational map $\mathbf{f}_\lambda$. We note that one cannot extend the above conjecture to actions of two arbitrary families $(\mathbf{f}_\lambda, \mathbf{g}_\lambda) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ (see the family of counterexamples from [**15**]). On the other hand, in the situation of families of Lattès maps associated to multiplication-by-2 on the elliptic curves E_λ, it is interesting to see if it is possible to establish a Bogomolov-type result for the main theorems of Masser and Zannier from [**20**–**22**]. One may also phrase a 'Manin–Mumford'-type conjecture along the lines of Conjecture 2.3, which might hold for preperiodic points over $\mathbb{C}$ (where one cannot define a height function) rather than $\bar{\mathbb{Q}}$.

We present here a brief comparison of Theorem 2.1 with the results from [**13**]. First, our present method covers all the families of polynomials treated by the authors in [**13**]; however, it does not provide explicit relations between the starting points $\mathbf{c}_1$ and $\mathbf{c}_2$ as provided in [**13**] (see also [**1**]). The fact that in Theorems 1.1 and 2.1 we do not obtain explicit relations between the starting points for the iterations (as obtained in [**1**, **13**]) is due to the fact that the analytic uniformization using Bottcher's Theorem (see [**8**]) cannot be used for giving an explicit formula for the local canonical height at an archimedean place for a rational map (which is *not* totally ramified at infinity). However, using the results proved in this paper we are able to extend the main result of [**13**] to the case the family of polynomial $\mathbf{f}_\lambda$ is parameterized by $\lambda \in X(\bar{\mathbb{Q}})$ where X is an arbitrary curve. Moreover, combining an argument in [**2**] and our results, we also replace the technical conditions in [**13**, Theorem 2.3] by a much weaker hypothesis.

THEOREM 2.4. *With the notation as above for X, η, Y, $\mathbf{A}$, assume $\mathbf{f} \in \mathbf{A}[z]$ such that the leading coefficient of $\mathbf{f}$ is constant. As before, we let $\mathbf{f}_\lambda \in \bar{\mathbb{Q}}[z]$ be the specialization of $\mathbf{f}$ obtained by evaluating each coefficient of $\mathbf{f}$ at $\lambda \in Y(\bar{\mathbb{Q}})$. Let $\mathbf{b}, \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}$ be such that $(\mathbf{a}_i, \mathbf{b}) = \mathbf{A}$ for $i = 1, 2$. In addition, assume*

$$\deg(\mathbf{a}_i) - 2\deg(\mathbf{b}) > M, \tag{2.3}$$

where M is the largest degree (as a function in $\mathbf{A}$) of any coefficient of $\mathbf{f}$.

If there exists an infinite family of $\lambda_n \in Y(\bar{\mathbb{Q}})$ such that

$$\lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n}}(\mathbf{c}_1(\lambda_n)) = \lim_{n\to\infty} \hat{h}_{\mathbf{f}_{\lambda_n}}(\mathbf{c}_2(\lambda_n)) = 0,$$

then there exist $\mathbf{h} \in \mathbf{A}[z]$ and integers $k > 0$, $m, n \geqslant 0$ such that $\mathbf{h} \circ \mathbf{f}^k = \mathbf{f}^k \circ \mathbf{h}$ and $\mathbf{f}^n(\mathbf{c}_1) = \mathbf{h}(\mathbf{f}^m(\mathbf{c}_2))$. Furthermore, if $\mathbf{b}$ is not constant, then $n = m$ and $\mathbf{h}$ is a constant family of homotheties.

REMARKS 2.5. (1) It is immediate to see that the conclusion of Theorem 2.4 confirms the conclusion of Conjecture 2.3 since the $\bar{\mathbb{Q}}$-curve $(\mathbf{c}_1, \mathbf{c}_2)$ contained in the three-dimensional $\bar{\mathbb{Q}}$-variety $(\mathbb{P}^1 \times \mathbb{P}^1)(\bar{\mathbb{Q}}(Y))$ would be contained in the surface given by the equation $\mathbf{f}^n(X_1) = \mathbf{h}(\mathbf{f}^m(X_2))$, which is invariant under the action of $\mathbf{f}^k$ (where X_1 and X_2 are the two coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$).

(2) If $\mathbf{b}$ is constant in Theorem 2.4, then condition (2.3) would follow from the fact that $\mathbf{c}_i$ is not preperiodic for $\mathbf{f}$, assuming that $\mathbf{f}$ is not a constant family. Indeed, then we obtain that $\hat{h}_{\mathbf{f}}(\mathbf{c}_i) > 0$ (see, for example, [**4**]) and thus $\deg(\mathbf{f}^n(\mathbf{c}_i))$ is unbounded and thus, at the expense of replacing $\mathbf{c}_i$ by a suitable iterate under $\mathbf{f}$, we may assume that (2.3) is verified.

3. *Notation and preliminaries*

For any quasiprojective variety X endowed with an endomorphism Φ, we call a point $x \in X$ *preperiodic* if there exist two distinct nonnegative integers m and n such that $\Phi^m(x) = \Phi^n(x)$, where by Φ^i we always denote the ith iterate of the endomorphism Φ. If $x = \Phi^n(x)$ for some positive integer n, then x is a *periodic* point of *period* n.

Let K be a number field; we let Ω_K be the set of all absolute values of K which extend the (usual) absolute values of $\mathbb{Q}$. For each $v \in \Omega_K$, we let v_0 be the (unique) absolute value of $\mathbb{Q}$ such that $v|_{\mathbb{Q}} = v_0$ and we let $N_v := [K_v : \mathbb{Q}_{v_0}]$. Let L be a finite extension of K, and let Ω_L be the set of all absolute values of K which extend the absolute values in Ω_K. For each $w \in \Omega_L$ extending some $v \in \Omega_K$, we let $N_w := N_v \cdot [L_w : K_v]$. The (naive) Weil height of any point $x \in L$ is defined as

$$h(x) = \frac{1}{[L:K]} \sum_{w\in\Omega_L} N_w \cdot \log\max\{1, |x|_w\}.$$

We will use the notation $\log^+(z)$ for $\log\max\{1,z\}$ for any real number z.

For each number field L, there exists a product formula for all nonzero elements x of L, that is,

$$\prod_{w\in\Omega_K} |x|_w^{N_w} = 1.$$

We fix an algebraic closure $\bar{K}$ of K and let $v \in \Omega_K$. Let $\mathbb{C}_v$ be the completion of a fixed algebraic closure of the completion of $(K, |\cdot|_v)$. When v is an archimedean valuation, then $\mathbb{C}_v = \mathbb{C}$. We use the same notation $|\cdot|_v$ to denote the extension of the absolute value of $(K_v, |\cdot|_v)$ to $\mathbb{C}_v$ and we also fix an embedding of $\bar{K}$ into $\mathbb{C}_v$.

Let $P = [x_0 : \cdots : x_k] \in \mathbb{P}^k(\bar{K})$ be given and let $P^{[1]}, \ldots, P^{[\ell]}$ denote the $\mathrm{Gal}(\bar{K}/K)$-conjugates of P. We let $h_v(P) := \log(\max\{|x_0|_v, \ldots, |x_k|_v\})$. Recall that the Weil height of P is given as follows:

$$h(P) := \frac{1}{\ell}\sum_{i=1}^{\ell}\sum_{v\in\Omega_K} N_v h_v(P^{[i]}).$$

In this paper, we are primarily interested in points on the projective line ($k = 1$). We fix an affine coordinate z on $\mathbb{P}^1$ and use the identification $\mathbb{P}^1(F) = F \cup \{\infty\}$ for any field F. That is, a point $x \in F$ is identified with the point $P = [x : 1] \in \mathbb{P}^1(F)$. The Weil height of $P = [x : 1]$ is simply denoted by $h(x)$.

Let $f \in K(x)$ be any rational map of degree $d \geqslant 2$. In [7], Call and Silverman defined the *global canonical height* $\hat{h}_f(x)$ for each $x \in \bar{K}$ as

$$\hat{h}_f(x) = \lim_{n\to\infty} \frac{h(f^n(x))}{d^n}.$$

In addition, Call and Silverman proved that the global canonical height decomposes as a sum of the local canonical heights, that is,

$$\hat{h}_f(x) = \frac{1}{[K(x):K]} \sum_{\sigma:K\to\bar{K}} \sum_{v\in\Omega_K} N_v\, \hat{h}_{f,v}(x^\sigma), \tag{3.1}$$

where, for each $v \in \Omega_K$, the function $\hat{h}_{f,v}$ is the local canonical height associated to f. For the existence and functorial property of the local canonical height, see [7, Theorem 2.1]. Using Northcott's Theorem, one deduces that x is preperiodic for f if and only if $\hat{h}_f(x) = 0$. This last statement does not hold if K is a function field over a smaller field K_0 since $\hat{h}_f(x) = 0$ for all $x \in K_0$ if f is defined over K_0.

We define heights in function fields similarly (see [5, 19]). So, if F is a function field of a projective normal variety $\mathcal{V}$ defined over a field K, then we denote by Ω_F the set of all absolute values on F associated to the irreducible divisors of $\mathcal{V}$. Then there exist positive integers N_v (for each $v \in \Omega_F$) such that $\prod_{v \in \Omega_F} |x|_v^{N_v} = 1$ for each nonzero $x \in F$. Also, we define the Weil height of any $P := [x_0 : \cdots : x_n] \in \mathbb{P}^n(F)$ as

$$h(P) = \sum_{v \in \Omega_F} N_v \cdot \log(\max\{|x_0|_v, \ldots, |x_n|_v\}).$$

Following [7], we let the canonical height of P with respect to an endomorphism φ of $\mathbb{P}^n$ of degree $d \geqslant 2$ be

$$\hat{h}_\varphi(P) = \lim_{n \to \infty} \frac{h(\varphi^n(P))}{d^n}.$$

4. *Heights and metrized line bundles*

Let L be a line bundle on a nonsingular projective variety X over a number field K and let $|\cdot|_v$ be an absolute value on K. We say that $\|\cdot\|_v$ is a *metric* on L if

$$\|(\alpha s)(P)\|_v = |\alpha|_v \|s(P)\|_v$$

for any $P \in X(K_v)$ and any section s. We say that $\bar{L}$ is an *adelic metrized line bundle* over K if it is equipped with a metric $\|\cdot\|_v$ at each place v of K.

When $\|\cdot\|_v$ is smooth and v is archimedean, we can form the *curvature* $c_1(\bar{L})_v$ of $\|\cdot\|_v$ as

$$c_1(\bar{L})_v = \frac{\partial\bar{\partial}}{\pi i} \log \|\cdot\|_v$$

on $X(\mathbb{C})$. At the nonarchimedean places, Chambert-Loir [9] has constructed an analog of curvature on $X^{an}_{\mathbb{C}_v}$, using methods from Berkovich spaces, in the case where the metric on the line bundle is *algebraic* in the sense of being determined by the extension of L to a line bundle $\mathcal{L}$ on a model $\mathcal{X}$ for X over $\mathfrak{o}_K$, that is, where $\|s(P)\|_v$ is determined by the intersection of $\operatorname{div} s$ with the Zariski closure of P in $\mathcal{X}$ at the place v.

An adelic metrized line bundle $\bar{L}$ is said to be algebraic if there is a model $\mathcal{X}$ that induces the metric $\|\cdot\|_v$ at each nonarchimedean place. An adelic metrized line bundle $\bar{L}$ is said to be *semipositive* (see [28, 31]) if there is a family of algebraic adelic metrized line bundles $\bar{L}_n$ (with metrics denoted as $\|\cdot\|_{v,n}$) such that:

- (SP1) at each v, we have that $\log \|\cdot\|_{v,n}$ converges uniformly (over all of $X(K_v)$) to $\log \|\cdot\|_v$;
- (SP2) for each n and each archimedean v, the metric $\|\cdot\|_{v,n}$ is smooth and the curvature of $\|\cdot\|_{v,n}$ is nonnegative and
- (SP3) for all n, all nonarchimedean v, and any curve complete C on the model $\mathcal{X}_n$ determining the metric $\|\cdot\|_v$ on L_n, the line bundle $\mathcal{L}_n$ (described above) pulls back to a divisor of positive degree on C.

In this case, one can assign a curvature $c_1(\bar{L})_v$ to $\bar{L}$ at each place v by taking the limits of the curvatures of the metrics on $\bar{L}_n$.

For any semipositive line bundle on a nonsingular subvariety X and any subvariety Z of X, one can define a height $h_{\bar{\mathcal{L}}}(Z)$ (see [31]). In the case of points $x \in X(\bar{K})$, with $\mathrm{Gal}(\bar{K}/K)$-conjugates $x^{[1]}, \ldots, x^{[\ell]}$, for example, it is defined as

$$\frac{1}{\ell} \sum_{i=1}^{\ell} \sum_{v \in \Omega_K} -N_v \cdot \log \|s(x^{[i]})\|_v, \tag{4.1}$$

where s is a meromorphic section of L with support disjoint from the conjugates of x.

The following result states a fundamental equidistribution principle for points of small height on an adelic metrized line bundle that is pivotal for our proof.

THEOREM 4.1 [**28**, Theorem 3.1]. *Suppose that X is a projective variety of dimension n over a number field, and $\bar{L}$ is an adelic metrized line bundle over X such that L is ample and the adelic metric is semipositive. Let $\{x_m\}$ be an infinite sequence of algebraic points in $X(\bar{K})$ which is generic and small. Then, for any place v of K, the Galois orbits of the sequence $\{x_m\}$ are equidistributed in the analytic space $X^{an}_{\mathbb{C}_v}$ with respect to the probability measure $d\mu_v = c_1(\bar{L})^n_v / \deg_L(X)$.*

The next result we need can be stated for an individual metric $\|\cdot\|_v$, where $v \in \Omega_K$ and L is an ample line bundle on a variety X over K_v. Recall that a metric $\|\cdot\|_v$ is said to be semipositive (see [**28**, **31**]) for archimedean v when it is a uniform limit of smooth metrics meeting condition (2) above, and that it is semipositive for nonarchimedean v when it is a uniform limit of algebraic metrics meeting condition (3) above.

THEOREM 4.2 [**29**, Theorem 2.10]. *Let L be an ample line bundle over X, where X is a projective variety over K_v, and let $\|\cdot\|_{v,1}$ and $\|\cdot\|_{v,2}$ be two semipositive metrics on L. Then $c_1(L, \|\cdot\|_{v,1})^{\dim X} = c_1(L, \|\cdot\|_{v,2})^{\dim X}$ if and only if $\|\cdot\|_{v,1}/\|\cdot\|_{v,2}$ is a constant.*

Note that in the special case where X has dimension 1, Theorem 4.2 can be proved quite easily, since the condition $c_1(L, \|\cdot\|_{v,1}) = c_1(L, \|\cdot\|_{v,2})$ implies that $\log(\|\cdot\|_{v,1}/\|\cdot\|_{v,2})$ is harmonic and thus constant (see [**27**, Proposition 3.1.1 and Corollary 3.2.11] or [**3**, Proposition 7.15]).

Combining Theorems 4.1 and 4.2, we have the following result.

COROLLARY 4.3. *Let L be an ample line bundle on X and let $\bar{L}_1$ and $\bar{L}_2$ be two semipositive adelic metrized line bundles over a number field K, each consisting of metrics on the same line bundle L. Assume, for all but finitely many places v, there exists a section s and a point $P \in X(\bar{\mathbb{Q}})$ such that $\|s(P)\|_{v,1} = \|s(P)\|_{v,2}$. Let $\{x_m\}$ be an infinite sequence of algebraic points in $X(\bar{K})$ that are Zariski-dense in X. Suppose that*

$$\lim_{m\to\infty} h_{\bar{L}_1}(x_m) = \lim_{m\to\infty} h_{\bar{L}_2}(x_m) = h_{\bar{L}_1}(X) = h_{\bar{L}_2}(X) = 0. \tag{4.2}$$

Then $h_{\bar{L}_1}(z) = h_{\bar{L}_2}(z)$ for all $z \in X(\bar{K})$.

Proof. Note that this is implicit in [**29**, Section 3], but for completeness, we give a proof. By Theorem 4.1, the sequence $\{x_m\}$ equidistribute with respect to both $c_1(\bar{L}_1)^{n-1}_v / \deg_L(X)$ and $c_1(\bar{L}_2)^{n-1}_v / \deg_L(X)$. Therefore, those two measures are the same, so the metrics are proportional. Since $h_{\bar{L}_1}(z)$ and $h_{\bar{L}_2}(z)$ are computed by evaluating $-\log\|s\|_{v,1}$ and $-\log\|s\|_{v,2}$ at z, it follows that $h_{\bar{L}_1}$ and $h_{\bar{L}_2}$ differ by a constant C (note that according to the hypothesis, for all but finitely many places v, the two metrics coincide and therefore the sum of all the local constants corresponding to $\log\|s\|_{v,2} - \log\|s\|_{v,1}$ is finite). Since there is a sequence on which both converge to the same value, this constant C must be zero, that is, $h_{\bar{L}_1} = h_{\bar{L}_2}$. ☐

As an example of family of metrics on the line bundle $L = \mathcal{O}_{\mathbb{P}^k}(1)$ on the projective space $\mathbb{P}^k$, let $F_n : \mathbb{P}^k \to \mathbb{P}^k$ be a family of morphisms written with respect to some coordinates $[X_0 : \cdots : X_k]$:

$$F_n = [F_n^{[0]} : \cdots : F_n^{[k]}],$$

where each $F_n^{[i]}$ is a homogeneous polynomial of degree e_n in $X_0, \ldots, X_n$. Then one might hope to metrize L as follows. Let $s = a_0X_0 + \cdots + a_kX_k$ be a global section of L. Then, at each place v, we define

$$\|s(t_0, \ldots, t_k)\|_{v,n} = \frac{|a_0t_0 + \cdots + a_kt_k|_v}{\max\{|F_n^{[0]}(t_0, \ldots, t_k)|_v, \ldots, |F_n^{[k]}(t_0, \ldots, t_k)|_v\}^{1/e_n}}. \tag{4.3}$$

Thus, we work with pull-backs of the semipositive metric

$$\|s(t_0, \ldots, t_k)\|_{v,n} = \frac{|a_0t_0 + \cdots + a_kt_k|_v}{\max(|t_0|_v, \ldots, |t_k|_v)}.$$

(This is the metric that gives rise to the usual Weil height on $\mathbb{P}^k$.)

As long as the family of metrics $\|\cdot\|_{v,n}$ converges uniformly, their limit gives a semipositive metric on L.

5. *Family of rational maps and specializations*

In this section, we study a one-parameter family of rational maps. Several different height functions appear into the picture. We prove a specialization theorem for these heights in the family of rational maps in question. An important specialization theorem has been proved by Call and Silverman [**7**, Theorem 4.1]. Using the method described in Section 4, we are able to give more precise information contained in the specialization theorem. Now let K be a number field. We fix the following notation throughout the remaining of this paper.

NOTATION.

X = a smooth, absolutely irreducible projective curve over K;
η = a fixed K-rational point of X;
$Y = X \setminus \{\eta\}$;
$F = K(X)$, the field of rational function on X;
$\mathbf{A} = \Gamma(\mathcal{O}_X, Y) \subset F$, the ring of rational functions of X regular away from η;
u = a uniformizer of η (defined over K);
$\deg(\cdot) = -\operatorname{ord}_\eta(\cdot)$.

By the definition of the degree function, we have that $\deg(\mathbf{a}) \geqslant 0$ for all $\mathbf{a} \in \mathbf{A}$. Let $\mathbf{a} \in \mathbf{A}$ be such that $\deg(\mathbf{a}) = n$. Then the function $g_\mathbf{a} := \mathbf{a}u^n$ has no pole at η. We call the constant $g_\mathbf{a}(\eta)$ the *leading coefficient* of $\mathbf{a}$.

5.1. *Family of rational maps*

We consider a morphism

$$\mathbf{f} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

of degree $d \geqslant 2$ over F and write

$$\mathbf{f}(x) = \frac{P(x)}{Q(x)} \quad \text{where } P(x), Q(x) \in \mathbf{A}[x]$$

such that $\mathrm{GCD}(P, Q) = 1$ (P and Q are viewed as elements in $F[x]$). For ease of the notation, we put $d_P := \deg_x P(x)$ and $d_Q = \deg_x Q(x)$. For a point $\lambda \in Y$, we use the following convention:

$$P_\lambda(x) = \sum_{i=0}^{d_P} \mathbf{c}_{P,d_P-i}(\lambda)x^i, \quad Q_\lambda(x) = \sum_{j=0}^{d_Q} \mathbf{c}_{Q,d_Q-j}(\lambda)x^j$$

and

$$\mathbf{f}_\lambda(x) = \frac{P_\lambda(x)}{Q_\lambda(x)} \quad \text{whenever } \mathbf{f}_\lambda \text{ is well defined.}$$

Here, $\mathbf{c}_{P,i}, \mathbf{c}_{Q,j} \in \mathbf{A}$ are coefficients of $P(x)$ and $Q(x)$, respectively. Thus, $\mathbf{f}$ gives rise to a family of rational maps $\{\mathbf{f}_\lambda\}$ parameterized by points λ ranging over an affine open subset of X. In the following, we work under the hypothesis of Theorem 2.1. Equivalently, $\mathbf{f}$ satisfies the following conditions:

(1) $d_P \geqslant d_Q + 2$;
(2) the leading coefficients of both P and Q as polynomials in x are constant and
(3) the resultant $R(\mathbf{f}) \in \mathbf{A}$ of $P(x)$ and $Q(x)$ is also a constant in K^*.

Condition (1) yields that $d = \max\{d_P, d_Q\} = d_P$ and $s = d_P - d_Q \geqslant 2$. Note that when $\mathbf{f}(x)$ is a polynomial, we have $s = d$. In this case, we set $Q(x) = 1$ and $\mathbf{c}_{Q,0} = 1$. By abuse of the notation, we also write $P(X, Y)$ and $Q(X, Y)$ for the homogeneous polynomials (in X, Y) $Y^d P(X/Y)$, $Y^d Q(X/Y)$ and let

$$\mathcal{F} : \mathbb{P}^1 \to \mathbb{P}^1 \quad \mathcal{F}([X : Y]) = [P(X, Y) : Q(X, Y)]$$

be a (fixed) representation of the given map $\mathcal{F}$ in homogeneous coordinates of $\mathbb{P}^1$. Using (3) above, it follows that there exist polynomials $S, T, U, V \in \mathbf{A}[X, Y]$ homogeneous in variables X, Y and positive integer $t \geqslant d$ such that

$$SP + TQ = X^t \tag{5.1}$$

and

$$UP + VQ = Y^t, \tag{5.2}$$

where the homogeneous degrees (in X and Y) are

$$\deg S = \deg T = \deg U = \deg V = t - d.$$

Let $\mathbf{c} := \mathbf{a}/\mathbf{b}$ be a rational function on X, where $\mathbf{a}, \mathbf{b} \in \mathbf{A}$. We assume that the ideal

$$(\mathbf{a}, \mathbf{b}) = \mathbf{A} \tag{5.3}$$

that is, $\mathbf{a}$ and $\mathbf{b}$ are relatively prime.

Put

$$m_1 = \max_{1 \leqslant i \leqslant d_P} \left\{ \frac{\deg \mathbf{c}_{P,i}}{i} \right\} \quad \text{and} \quad m_2 = \max_{1 \leqslant j \leqslant d_Q} \left\{ \frac{\deg \mathbf{c}_{Q,j}}{j} \right\}, \tag{5.4}$$

and $m = m_1 + m_2$. By convention, we set $m_2 = 0$ in the case where $\mathbf{f}$ is a polynomial map. The degree function deg on $\mathbf{A}$ has a natural extension to F given by $\deg(\mathbf{c}) = \deg(\mathbf{a}) - \deg(\mathbf{b})$ for $\mathbf{c} \in F$ given above. In this section and the next, we assume that $\mathbf{c}$ satisfies

$$\deg(\mathbf{c}) > m. \tag{5.5}$$

In particular, we know $\deg(\mathbf{c}) \geqslant 1$ (since $m \geqslant 0$). In the following, we also set $d_{\mathbf{a}} := \deg(\mathbf{a})$, $d_{\mathbf{b}} := \deg(\mathbf{b})$ and $d_{\mathbf{c}} := \deg(\mathbf{c})$.

For each integer $n \geqslant 0$, we let $A_{\mathbf{c},n}$ and $B_{\mathbf{c},n}$ be elements in $\mathbf{A}$ defined recursively as follows:

$$A_{\mathbf{c},0} = \mathbf{a} \quad \text{and} \quad B_{\mathbf{c},0} = \mathbf{b},$$

while, for all $n \geqslant 0$ we have

$$A_{\mathbf{c},n+1} = P(A_{\mathbf{c},n}, B_{\mathbf{c},n}) \tag{5.6}$$

and

$$B_{\mathbf{c},n+1} = Q(A_{\mathbf{c},n}, B_{\mathbf{c},n}). \tag{5.7}$$

PROPOSITION 5.1. *For all $n \geqslant 1$, we have $(A_{\mathbf{c},n}, B_{\mathbf{c},n}) = \mathbf{A}$ and $\deg(A_{\mathbf{c},n}) = d_{\mathbf{a}} \cdot d^n$ while $\deg(B_{\mathbf{c},n}) = d_{\mathbf{a}} d^n - d_{\mathbf{c}} s^n$. Furthermore, the leading coefficient of $A_{\mathbf{c},n}$ is $c_{P,0}^{(d^n-1)/(d-1)} \cdot c_{\mathbf{a}}^{d^n}$, where $c_{\mathbf{a}}$ is the leading coefficient of $\mathbf{a}$.*

Proof. The computation of degrees is straightforward using (5.5). The computation for the leading coefficient of $A_{\mathbf{c},n}$ follows from the definition of the leading coefficient and induction on n.

We show next that $A_{\mathbf{c},n}$ and $B_{\mathbf{c},n}$ are relatively prime polynomials. This assertion follows easily by induction on n; the case $n = 0$ is immediate.

Assume that $A_{\mathbf{c},n}$ and $B_{\mathbf{c},n}$ are relatively prime for some $n \geqslant 0$ and we prove that $A_{\mathbf{c},n+1}$ and $B_{\mathbf{c},n+1}$ are also relatively prime. Substitute $X = A_{\mathbf{c},n}$ and $Y = B_{\mathbf{c},n}$ into (5.1) and (5.2) and by the recursive relations (5.6) and (5.7), we have

$$S(A_{\mathbf{c},n}, B_{\mathbf{c},n})A_{\mathbf{c},n+1} + T(A_{\mathbf{c},n}, B_{\mathbf{c},n})B_{\mathbf{c},n+1} = A_{\mathbf{c},n}^t$$

and

$$U(A_{\mathbf{c},n}, B_{\mathbf{c},n})A_{\mathbf{c},n+1} + V(A_{\mathbf{c},n}, B_{\mathbf{c},n})B_{\mathbf{c},n+1} = B_{\mathbf{c},n}^t.$$

It follows that the ideals $(A_{\mathbf{c},n}^t, B_{\mathbf{c},n}^t) \subseteq (A_{\mathbf{c},n+1}, B_{\mathbf{c},n+1})$. Since by the induction hypothesis, we have $(A_{\mathbf{c},n}, B_{\mathbf{c},n}) = \mathbf{A}$, we conclude $(A_{\mathbf{c},n+1}, B_{\mathbf{c},n+1}) = \mathbf{A}$. This completes the induction and the proof of Proposition 5.1. □

REMARK 5.2. Note that by the definition of the iterates $\mathbf{f}^n$, we have $\mathbf{f}^n(\mathbf{c}) = A_{\mathbf{c},n}/B_{\mathbf{c},n}$, and thus Proposition 5.1 yields that $\mathbf{c}$ is not preperiodic under the action of $\mathbf{f}$.

The following is an easy corollary of Proposition 5.1.

COROLLARY 5.3. *With the above notation, if $\mathbf{c}(\lambda)$ is preperiodic under the action of $\mathbf{f}_\lambda$, then $\lambda \in Y(\bar{K})$.*

Proof. If $\mathbf{c}(\lambda)$ is preperiodic under $\mathbf{f}_\lambda$, then, for some positive integers $n > m$, we have $\mathbf{f}_\lambda^n(\mathbf{c}(\lambda)) = \mathbf{f}_\lambda^m(\mathbf{c}(\lambda))$. So,

$$\frac{A_{\mathbf{c},n}(\lambda)}{B_{\mathbf{c},n}(\lambda)} = \frac{A_{\mathbf{c},m}(\lambda)}{B_{\mathbf{c},m}(\lambda)},$$

where $A_{\mathbf{c},n}$ and $B_{\mathbf{c},n}$, and also $A_{\mathbf{c},m}$ and $B_{\mathbf{c},m}$ are relatively prime. Thus λ is a zero of the nonzero rational function $A_{\mathbf{c},n}B_{\mathbf{c},m} - A_{\mathbf{c},m}B_{\mathbf{c},n}$ over K, and hence $\lambda \in Y(\bar{K})$. □

5.2. *Specialization theorem*

For a given rational map $\mathbf{f}$ of degree $d \geqslant 2$ over F and a point $\mathbf{c} \in \mathbb{P}^1(F)$, there are three heights: $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$, $\hat{h}_{\mathbf{f}}(\mathbf{c})$ and $h(\lambda)$. Namely, given $\lambda \in X(\bar{K})$ such that $\mathbf{f}_\lambda$ is a well-defined rational map $\mathbf{f}_\lambda$ over $K(\lambda)$, the height $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))$ is the canonical height of $\mathbf{c}(\lambda)$ associated to $\mathbf{f}_\lambda$ and $h(\lambda)$ is a Weil height associated to a degree 1 divisor class on X, while $\hat{h}_{\mathbf{f}}(\mathbf{c})$ is the canonial height of $\mathbf{c}$ associated to $\mathbf{f}$ over F. Call and Silverman [**7**, Theorem 4.1] have shown that

$$\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = \hat{h}_{\mathbf{f}}(\mathbf{c})h(\lambda) + o(h(\lambda)), \tag{5.8}$$

which generalizes a result of Silverman [**24**] on heights of families of abelian varieties. In a recent paper [**18**], Ingram shows that, for a family of polynomial maps $\mathbf{f} \in F[x]$ and $P \in \mathbb{P}^1(F)$, there is a divisor $D = D(\mathbf{f}, P) \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ of degree $\hat{h}_{\mathbf{f}}(P)$ such that

$$\hat{h}_{\mathbf{f}_\lambda}(P_\lambda) = h_D(\lambda) + O(1).$$

This result is an analog of Tate's theorem [**26**] in the setting of arithmetic dynamics. Using this result and applying an observation of Lang [**19**, Chapter 5, Proposition 5.4], the error term in (5.8) is improved to $O(h(\lambda)^{1/2})$ and, furthermore, in the special case where $X = \mathbb{P}^1$ the error term can be replaced by $O(1)$ (see [**18**, Corollary 2]).

In order to apply Theorem 4.2 to our situation, the error term in (5.8) needs to be controlled within $O(1)$. Ingram's result shows that this is true if $\mathbf{f}$ is a polynomial map and the parameter space $X = \mathbb{P}^1$. In this paper, we provide another set of conditions for $\mathbf{f}$ and the point $P \in \mathbb{P}^1(F)$ so that the error $o(h(\lambda))$ can be replaced by $O(1)$ for the height function with respect to some degree 1 divisor class on X.

THEOREM 5.4. *Let $\mathbf{f}(x) := P(x)/Q(x) \in F(x)$ be of degree $d \geqslant 2$ over F and assume that $\mathbf{f}$ satisfies the following conditions.*

(1) *The resultant $R(\mathbf{f})$ and the leading coefficients of $P(x)$ and $Q(x)$ are nonzero constants.*
(2) *The point $x = \infty$ is a superattracting periodic point for $\mathbf{f}$.*

Let $\mathbf{c} \in \mathbb{P}^1(F)$ be such that the sequence $\{\deg(\mathbf{f}^n(\mathbf{c}))\}_{n \geqslant 0}$ is unbounded. Then for $\lambda \in Y(\bar{K})$, we have

$$\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) = \hat{h}_{\mathbf{f}}(\mathbf{c})h(\lambda) + O(1),$$

where h is a height function associated to the divisor class containing the divisor η.

REMARK 5.5. (1) Note that on an algebraic curve of genus at least 1, two divisor classes of the same degree are algebraically equivalent. If in Theorem 5.4 the genus of the parameter space X is positive and h is a height function associated to an (arbitrary) divisor class of degree 1, then by [**19**, Chapter 5, Proposition 5.4] again we also conclude that the error is $O(h(\lambda)^{1/2})$. On the other hand, if $X = \mathbb{P}^1$, then the error is still $O(1)$ for any height function h associated to a degree 1 divisor class on $\mathbb{P}^1$. Furthermore, as a corollary to our result, an analog of Tate's theorem [**26**] can also be generalized for rational maps $\mathbf{f}$ and points $\mathbf{c}$ satisfying conditions given in Theorem 5.4.

(2) We actually show that the function $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))/\hat{h}_{\mathbf{f}}(\mathbf{c})$ is a height function coming from a semipositive metrized line bundle on X. In the case proved by Ingram that is not covered by our theorem, that is, the case where $\mathbf{f}$ is a polynomial with parameter space $X = \mathbb{P}^1$ and $\mathbf{c} \in \mathbb{P}^1(F)$ without any further restriction, it would be interesting to see whether or not $\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))/\hat{h}_{\mathbf{f}}(\mathbf{c})$ also gives rise to a height function coming from a semipositive metrized line bundle on $\mathbb{P}^1$.

(3) We do not know for general $\mathbf{f}$ and $\mathbf{c}$ whether or not there always exists a height function h on X associated to some divisor class (depending on $\mathbf{f}$ and $\mathbf{c}$) of degree 1 such that the error term in (5.8) is $O(1)$ (we are indebted to Silverman for raising this question). It would be interesting to study this question to see if there is an affirmative answer.

Theorem 5.4 will follow from Proposition 7.3 proved below. The proof of Theorem 5.4 follows the idea described in Section 4 and will be given later. The following two sections are devoted to the proof of Proposition 7.3.

6. *Growth of the iterates in fibers above X*

We continue with the notation from the previous section.

Recall that we have fixed a uniformizer u of η. Then there exists a Zariski open neighborhood Z of η such that the uniformizer u is a regular function on Z. We fix such a neighborhood of η. For a given place $v \in \Omega_K$, $Y(\mathbb{C}_v)$ has a topology called the v-adic topology induced by the absolute value $|\cdot|_v$ on $\mathbb{C}_v$. Each $\mathbf{a} \in \mathbf{A}$ yields a continuous function $\mathbf{a}: Y(\mathbb{C}_v) \to \mathbb{C}_v$ with respect to the v-adic topology. For any large $L > 0$, we let $V_{L,v} \subset Z(\mathbb{C}_v)$ be the v-adic open neighborhood of η containing all points $\lambda \in Z(\mathbb{C}_v)$ such that $|u(\lambda)|_v < 1/L$. If there is no danger of confusion, then we drop the subscript v below. Denote the complement of V_L by $U_L := X(\mathbb{C}_v) \setminus V_L \subset Y(\mathbb{C}_v)$. It follows that $\mathbf{a}$ is bounded on U_L. Let $n = \deg(\mathbf{a})$ and put $g_{\mathbf{a}} = \mathbf{a}u^n$. By increasing L if necessary, we may assume that $g_{\mathbf{a}}$ is bounded on V_L; let $C > 0$ be an upper bound for $|g_{\mathbf{a}}|_v$ on V_L. Thus for each λ in the boundary of V_L, we have that $|\mathbf{a}(\lambda)|_v \leqslant CL^n$. Furthermore, for L sufficiently large, the maximum of $|\mathbf{a}(\lambda)|_v$ on U_L is attained on the boundary of U_L (which is the same as the boundary of V_L) and thus $|\mathbf{a}(\lambda)|_v \leqslant CL^n$ for all $\lambda \in U_L$. Note that even though *a priori*, C depends on L (and also on $\mathbf{a}$ and v), the dependence on L is not essential since once we replace L with a larger number L', the same value of C would work as an upper bound for $g_{\mathbf{a}}$ on $V_{L'}$ because $V_{L'} \subset V_L$ (this fact will be used in our proof). More generally, for a nonempty finite subset T of $\mathbf{A}$ and large L, by shrinking V_L if necessary, we may assume that there exists a positive constant C depending only on T and v such that the inequality $|\mathbf{a}(\lambda)|_v \leqslant CL^n$ holds for all $\mathbf{a} \in T$ and $\lambda \in U_L$ where $n = \max_{\mathbf{a}\in T} \deg(\mathbf{a})$. Furthermore, for any polynomial $g(x) \in \mathbb{C}_v[x]$ we define $|g|_v$ to be the maximum of the v-adic norms of its coefficients.

Proposition 6.1. *Let $v \in \Omega_K$ be any place, and let*

$$M_n(\lambda) := \max\{|A_{\mathbf{c},n}(\lambda)|_v, |B_{\mathbf{c},n}(\lambda)|_v\}$$

for each $n \geqslant 0$ and $\lambda \in Y(\mathbb{C}_v)$. Let $L \geqslant 1$ be a large positive number and let $U_L \subset Y(\mathbb{C}_v)$ be determined as above. Then there exist positive constants C_1, C_2 depending only on v, L and on the coefficients $\mathbf{c}_{P,i}$ of P and $\mathbf{c}_{Q,j}$ of Q such that, for all $n \geqslant 0$, we have

$$C_1 M_n(\lambda)^d \leqslant M_{n+1}(\lambda) \leqslant C_2 M_n(\lambda)^d$$

for all $\lambda \in U_L$.

Proof. Let $\lambda \in U_L$ be given. The proof uses standard techniques in height theory. Before we deduce the upper bound in the above inequalities, we first note

$$|P_\lambda|_v = \max\{|c_{P,i}(\lambda)|_v : i = 0, \dots, d_P\} \quad \text{and} \quad |Q_\lambda|_v = \max\{|c_{Q,j}(\lambda)|_v : j = 0, \dots, d_Q\}.$$

Therefore, there exists a constant C_3 depending only on L and on the coefficients of P_λ such that $|P_\lambda|_v \leqslant C_3 L^{m_1 d_P}$ and $|Q_\lambda|_v \leqslant C_3 L^{m_2 d_Q}$ for all $\lambda \in U_L$. For any integer k, we use the following notation:

$$\epsilon_v(k) = \begin{cases} k & \text{if } v \text{ is archimedean,} \\ 1 & \text{if } v \text{ is nonarchimedean.} \end{cases}$$

By (5.6) and (5.7), we have

$$\begin{aligned} |A_{\mathbf{c},n+1}(\lambda)|_v &= |P_\lambda(A_{\mathbf{c},n}(\lambda), B_{\mathbf{c},n}(\lambda))|_v \\ &\leqslant (\epsilon_v(d_P+1)|P_\lambda|_v) M_n(\lambda)^{d_P} \\ &\leqslant (\epsilon_v(d_P+1) C_3 L^{m_1 d_P}) M_n(\lambda)^{d_P} \end{aligned}$$

and

$$\begin{aligned} |B_{\mathbf{c},n+1}(\lambda)|_v &\leqslant |B_{\mathbf{c},n}(\lambda)^s Q_\lambda(A_{\mathbf{c},n}(\lambda), B_{\mathbf{c},n}(\lambda))|_v \\ &\leqslant (\epsilon_v(d_Q+1) C_3 L^{m_2 d_Q}) M_n(\lambda)^{d_P}. \end{aligned}$$

So, the right-hand side inequality from the conclusion of Proposition 6.1 holds with $C_2 = \epsilon_v(d_P+1)C_3(L^{m_1 d_P}+L^{m_2 d_Q})$, for example.

Next, we deduce a complementary inequality. For this, we substitute $X = A_{\mathbf{c},n}$ and $Y = B_{\mathbf{c},n}$ into (5.1) and (5.2). Then, as in the proof of Proposition 5.1 we have

$$S(A_{\mathbf{c},n}, B_{\mathbf{c},n})A_{\mathbf{c},n+1} + T(A_{\mathbf{c},n}, B_{\mathbf{c},n})B_{\mathbf{c},n+1} = A^t_{\mathbf{c},n}$$

and

$$U(A_{\mathbf{c},n}, B_{\mathbf{c},n})A_{\mathbf{c},n+1} + V(A_{\mathbf{c},n}, B_{\mathbf{c},n})B_{\mathbf{c},n+1} = B^t_{\mathbf{c},n}.$$

Note that, as polynomials in variables X and Y, the coefficients of S, T, U and V are in $\mathbf{A}$. Let the maximal degrees of coefficients of S, T, U and V be ℓ. Then, for $\lambda \in U_L$, there exists a positive real constant C_4 such that

$$\max\{|S|_v, |T|_v, |U|_v, |V|_v\} \leqslant C_4 L^{\ell}.$$

Applying the triangle inequality, we have

$$\begin{aligned}|A_{\mathbf{c},n}(\lambda)|^t_v &\leqslant \epsilon_v(t-d_P+1)|S|_v M_n(\lambda)^{t-d_P}M_{n+1}(\lambda) + \epsilon_v(t-d_P+1)|T|_v M_n(\lambda)^{t-d_P}M_{n+1}(\lambda)\\ &\leqslant 2\epsilon_v(t-d_P+1)C_4L^{\ell}M_n(\lambda)^{t-d_P}M_{n+1}(\lambda)\end{aligned}$$

and, similarly,

$$|B_{c,n}(\lambda)|^t_v \leqslant 2\epsilon_v(t-d_P+1)C_4L^{\ell}M_n(\lambda)^{t-d_P}M_{n+1}(\lambda).$$

Hence,

$$M_n(\lambda)^t \leqslant 2\epsilon_v(t-d_P+1)C_4L^{\ell}M_n(\lambda)^{t-d_P}M_{n+1}(\lambda)$$

and thus the desired lower bound from Proposition 6.1 is obtained by taking $C_1 = 1/(2\epsilon_v(t-d_P+1)C_4L^{\ell})$ and note $d_P = d$. □

Next, we fix $v \in \Omega_K$ and show that $M_n(\lambda)$ also satisfies similar relations as stated in Proposition 6.1 for $\lambda \in V_L$ with L large enough. In the following, the notation $v\text{-}\lim_{\lambda\to\eta}$ means that the limit is taken for the point λ approaching η with respect to the v-adic topology. We first observe

$$v\text{-}\lim_{\lambda\to\eta}\frac{|P_\lambda(\mathbf{c}(\lambda))|_v}{|\mathbf{c}(\lambda)|_v^{d_P}} = |c_{P,0}|_v \quad\text{and}\quad v\text{-}\lim_{\lambda\to\eta}\frac{|Q_\lambda(\mathbf{c}(\lambda))|_v}{|\mathbf{c}(\lambda)|_v^{d_Q}} = |c_{Q,0}|_v.$$

Indeed, the assertions follow from the choice of $d_{\mathbf{c}} = \deg \mathbf{c}$ such that $id_{\mathbf{c}} > \deg c_{P,i}$ for $i = 1, \ldots, d_P$ and $jd_{\mathbf{c}} > \deg c_{Q,j}$ for $j = 1, \ldots, d_Q$. Furthermore, we have $v\text{-}\lim_{\lambda\to\eta}|\mathbf{c}(\lambda)|_v|u(\lambda)|_v^{d_{\mathbf{c}}} = |c_{\mathbf{a}}/c_{\mathbf{b}}|_v$ and $v\text{-}\lim_{\lambda\to\eta}|f_\lambda(\mathbf{c}(\lambda))|_v/|\mathbf{c}(\lambda)|^s_v = |c_{P,0}/c_{Q,0}|_v$. It follows that there exist positive real numbers $L_1 > 1$ and $\delta_1 < 1$ such that, for all $\lambda \in V_{L_1}\setminus\{\eta\}$, we have

$$\delta_1|\mathbf{c}(\lambda)|^s_v \leqslant |f_\lambda(\mathbf{c}(\lambda))|_v \leqslant \frac{1}{\delta_1}|\mathbf{c}(\lambda)|^s_v, \tag{6.1}$$

$$\delta_1|z(\lambda)|^{d_{\mathbf{c}}}_v \leqslant |\mathbf{c}(\lambda)|_v \leqslant \frac{1}{\delta_1}|z(\lambda)|^{d_{\mathbf{c}}}_v, \tag{6.2}$$

where, for convenience, we set $z = 1/u$ so that $z(\lambda) = 1/u(\lambda)$ for $\lambda \in V_L \setminus \{\eta\}$. Furthermore, without loss of generality, we may assume $\delta_1 < |c_{P,0}|_v/2|c_{Q,0}|_v$.

Lemma 6.2. *Let $L_2 \geqslant L_1$ be a real number. Then there exists a real number $L_3 \geqslant L_2$ such that, for all $\mathbf{x} \in F$ satisfying $|\mathbf{x}(\lambda)/\mathbf{c}(\lambda)|_v > 2$ for all $\lambda \in V_{L_2} \setminus \{\eta\}$, we have*

$$|f_\lambda(\mathbf{x}(\lambda))|_v > \delta_1 |\mathbf{x}(\lambda)|_v^s$$

for all $\lambda \in V_{L_3} \setminus \{\eta\}$.

Proof. Let the Laurent series expansion in x^{-1} of $f_\lambda(x)$ be as follows:

$$f_\lambda(x) = \frac{P_\lambda(x)}{Q_\lambda(x)} = \sum_{k \geqslant 0} \alpha_k x^{s-k},$$

where $\alpha_k \in \mathbf{A}$ and $\alpha_0(\lambda) = c_{P,0}/c_{Q,0} \in K^*$. We estimate next $|\alpha_k(\lambda)|_v$ for $\lambda \in V_{L_2} \setminus \{\eta\}$ as k varies. For this, we write $x^{d_Q} Q_\lambda(x)^{-1} = \sum_{j \geqslant 0} \beta_j x^{-j} \in \mathbf{A}[[x^{-1}]]$ and we claim that there exist positive real numbers C_5 and C_6 such that

$$|\beta_j(\lambda)|_v \leqslant C_5 (C_6 |z(\lambda)|_v^{m_2})^j. \tag{6.3}$$

Indeed, (6.3) follows from the fact that $\beta_0 = 1/c_{Q,0}$ while for each $j \geqslant 1$, we have

$$-c_{Q,0}\beta_j = \sum_{\substack{i_1+i_2=j \\ 1 \leqslant i_1 \leqslant d_Q \\ 0 \leqslant i_2 \leqslant j-1}} c_{Q,i_1} \beta_{i_2}.$$

So, using $|c_{Q,i}(\lambda)|_v = O(|z(\lambda)|_v^{m_2 i})$, an easy induction completes the proof of (6.3). On the other hand, we have

$$\alpha_k(\lambda) = \sum_{\substack{i_1+i_2=k \\ 0 \leqslant i_1 \leqslant d_P \\ 0 \leqslant i_2 \leqslant d_Q}} \beta_{i_2}(\lambda) c_{P,i_1}(\lambda).$$

In particular, $\alpha_0 = c_{P,0}/c_{Q,0}$. Using (6.3) coupled with the fact that $|c_{P,i}(\lambda)|_v = O(|z(\lambda)|_v^{m_1 i})$, we obtain that there exist positive real numbers C_7 and C_8 (independent of k) such that

$$|\alpha_k(\lambda)|_v \leqslant C_7 (C_8 |z(\lambda)|_v^{m_1+m_2})^k \quad \text{for all } k.$$

Since $|\mathbf{c}(\lambda)|_v \geqslant \delta_v |z(\lambda)|_v^{d_\mathbf{c}}$ and $d_\mathbf{c} > m = m_1 + m_2$, we obtain that, for sufficiently large $|z(\lambda)|_v$, we have $\max\{|\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}|_v : k \in \mathbb{N}\} < |\alpha_0|_v/2$ and, furthermore, if v is archimedean, then $\sum_{k \geqslant 1} |\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}|_v^2$ is convergent and bounded above by $|\alpha_0|_v^2/4$.

We let $L_3 > L_2$ be a sufficiently large real number such that, for $|z(\lambda)|_v > L_3$, we have

$$\sum_{k \geqslant 1} |\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}|_v^2 < |\alpha_0|_v^2/4 \quad \text{if } v \text{ is archimedean}$$

and

$$\max\{|\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}|_v : k \geqslant 1\} < \frac{|\alpha_0|_v}{2} \quad \text{if } v \text{ is nonarchimedean.}$$

Now let $\lambda \in V_{L_3} \setminus \{\eta\}$ and $\mathbf{x} \in F$ such that $|\mathbf{x}(\lambda)/\mathbf{c}(\lambda)|_v > 2$. Write

$$\sum_{k \geqslant 1} \alpha_k(\lambda)\mathbf{x}(\lambda)^{-k} = \sum_{k \geqslant 1} \alpha_k(\lambda)\mathbf{c}(\lambda)^{-k} \left(\frac{\mathbf{x}(\lambda)}{\mathbf{c}(\lambda)}\right)^{-k}.$$

If v is nonarchimedean, then we have

$$\left|\sum_{k \geqslant 1} \alpha_k(\lambda)\mathbf{x}(\lambda)^{-k}\right|_v \leqslant \max\left\{\left|\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}\left(\frac{\mathbf{x}(\lambda)}{\mathbf{c}(\lambda)}\right)^{-k}\right|_v : k \geqslant 1\right\} \leqslant \frac{|\alpha_0|_v}{2}.$$

Therefore,

$$\left|\frac{f_\lambda(\mathbf{x}(\lambda))}{\mathbf{x}(\lambda)^s}\right|_v = \left|\alpha_0 + \sum_{k\geqslant 1}\alpha_k(\lambda)\mathbf{x}(\lambda)^{-k}\right|_v = |\alpha_0|_v \geqslant \delta_1.$$

Now assume that v is archimedean. By the choice of L_3, the two sequences of complex numbers $(\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k})_{k\geqslant 1}$ and $((\mathbf{x}(\lambda)/\mathbf{c}(\lambda))^{-k})_{k\geqslant 1}$ are both square summable for each $\lambda \in V_{L_3} \setminus \{\eta\}$. Hence, by the Cauchy–Schwartz inequality we see

$$\begin{aligned}\left|\sum_{k\geqslant 1}\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}\left(\frac{\mathbf{x}(\lambda)}{\mathbf{c}(\lambda)}\right)^{-k}\right|_v^2 &\leqslant \left(\sum_{k\geqslant 1}|\alpha_k(\lambda)\mathbf{c}(\lambda)^{-k}|_v^2\right)\left(\sum_{k\geqslant 1}\left|\frac{\mathbf{x}(\lambda)}{\mathbf{c}(\lambda)}\right|_v^{-2k}\right)\\ &\leqslant \frac{|\alpha_0|_v^2}{4}\cdot\left(\sum_{k\geqslant 1}\frac{1}{4^k}\right) \leqslant \frac{|\alpha_0|_v^2}{4}.\end{aligned}$$

Hence,

$$\left|\frac{f_\lambda(\mathbf{x}(\lambda))}{\mathbf{x}(\lambda)^s}\right|_v \geqslant \left||\alpha_0|_v - \sqrt{\frac{|\alpha_0|_v^2}{4}}\right| \geqslant \frac{|\alpha_0|_v}{2}.$$

In both cases, we have shown that for all $\lambda \in V_{L_3}$ we have $|f_\lambda(\mathbf{x}(\lambda))|_v \geqslant \delta_1|\mathbf{x}(\lambda)|_v^s$, as desired. □

PROPOSITION 6.3. *There exists a number $L_3 \geqslant L_1$ depending only on the coefficients of P_λ, Q_λ (and on L_1) such that for all $n \in \mathbb{N}$ and all $\lambda \in V_{L_3} \setminus \{\eta\}$, we have*

$$|f_\lambda^n(\mathbf{c}(\lambda))|_v \geqslant \delta_1^{(s^n-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^n} \geqslant \delta_1^{(s^{n+1}-1)/(s-1)}|z(\lambda)|_v^{d_{\mathbf{c}}s^n}. \tag{6.4}$$

Proof. First, we note that if $|z(\lambda)|_v > L_3 \geqslant L_1$, then (6.2) yields the second inequality from (6.4). So, we are left to prove the first inequality in (6.4).

We claim that there exists a real number L_2 larger than L_1 which also satisfies the following properties:

(a) if $\lambda \in V_{L_2} \setminus \{\eta\}$, then $\delta_1|\mathbf{c}(\lambda)|_v^{s-1} > 2$;
(b) if $\lambda \in V_{L_2} \setminus \{\eta\}$, then $|f_\lambda(\mathbf{c}(\lambda))|_v \geqslant \delta_1|\mathbf{c}(\lambda)|_v^s$.

We can obtain inequality (a) above since if $|z(\lambda)|_v > L_2 \geqslant L_1$, then

$$|\mathbf{c}(\lambda)|_v \geqslant \delta_1|z(\lambda)|_v^{d_{\mathbf{c}}} > \delta_1 L_2^{d_{\mathbf{c}}}, \tag{6.5}$$

and thus if $L_2 > (2/\delta_1^s)^{1/d_{\mathbf{c}}(s-1)}$, inequality (a) is satisfied. Inequality (b) follows immediately from (6.1) since $L_2 \geqslant L_1$.

We let $L_3 \geqslant L_2$ be the real number satisfying the conclusion of Lemma 6.2. Let $\lambda \in V_{L_3} \setminus \{\eta\}$. The proof of the first inequality in the conclusion of Proposition 6.3 is by induction on $n \geqslant 1$. The inequality (6.4) for $n = 1$ is precisely inequality (b) above.

Next we prove the inductive step; so we assume that (6.4) holds for some $n \geqslant 1$ and we will prove

$$|f_\lambda^{n+1}(\mathbf{c}(\lambda))|_v \geqslant \delta_1^{(s^{n+1}-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^{n+1}}.$$

By induction hypothesis, we know

$$|f_\lambda^n(\mathbf{c}(\lambda))|_v \geqslant \delta_1^{(s^n-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^n} \geqslant \delta_1^{(s^{n+1}-1)/(s-1)}|z(\lambda)|_v^{d_{\mathbf{c}}s^n}.$$

We shall apply Lemma 6.2 to $\mathbf{x}(\lambda) = f_\lambda^n(\mathbf{c}(\lambda))$. In order to do this, we need to check

$$|f_\lambda^n(\mathbf{c}(\lambda))/\mathbf{c}(\lambda)|_v > 2 \quad \text{if } \lambda \in V_{L_2} \setminus \{\eta\}. \tag{6.6}$$

Indeed, we note

$$\begin{aligned}\left|\frac{f_\lambda^n(\mathbf{c}(\lambda))}{\mathbf{c}(\lambda)}\right|_v &\geqslant \delta_1^{(s^n-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^n-1} \quad \text{by the inductive hypothesis}\\ &\geqslant (\delta_1|\mathbf{c}(\lambda)|_v^{s-1})^{(s^n-1)/(s-1)}\\ &> 2^{(s^n-1)/(s-1)} \quad \text{by inequality (a) above}\\ &\geqslant 2 \quad \text{since } s \geqslant 2.\end{aligned}$$

Now, by Lemma 6.2 applied to $\mathbf{x}(\lambda) = f_\lambda^n(\mathbf{c}(\lambda))$, we have

$$\begin{aligned}|f_\lambda^{n+1}(\mathbf{c}(\lambda))|_v &= |f_\lambda(f_\lambda^n(\mathbf{c}(\lambda)))|_v\\ &\geqslant \delta_1|f_\lambda^n(\mathbf{c}(\lambda))|_v^s \quad \text{by Lemma 6.2}\\ &\geqslant \delta_1(\delta_1^{(s^n-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^n})^s \quad \text{by induction hypothesis,}\\ &= \delta_1^{(s^{n+1}-1)/(s-1)}|\mathbf{c}(\lambda)|_v^{s^{n+1}}.\end{aligned}$$

This concludes the inductive step and the proof of Proposition 6.3. □

REMARK 6.4. Let L_3 be the constant for the place v as chosen in Proposition 6.3. As a corollary, we have that $\mathbf{c}(\lambda)$ is not preperiodic for $\mathbf{f}_\lambda$ for $\lambda \in V_{L_3} \setminus \{\eta\}$.

PROPOSITION 6.5. *There exist real numbers $L_4 \geqslant 1$, $C_9 > 0$ and $C_{10} > 0$ such that, for all $\lambda \in V_{L_4} \setminus \{\eta\}$, we have*

$$C_9 M_n(\lambda)^d \leqslant M_{n+1}(\lambda) \leqslant C_{10} M_n(\lambda)^d$$

for all $n \in \mathbb{N}$.

Proof. We let L_3 be defined as in Proposition 6.3 and let $L_5 \geqslant L_3$ satisfy also the inequality

$$\delta_1^{s+1} L_5^{d_{\mathbf{c}} s} > L_5^{d_{\mathbf{c}}}, \tag{6.7}$$

or equivalently $L_5 > \delta_1^{-(s+1)/(d_{\mathbf{c}}(s-1))}$.

We first claim that L_5 satisfies the following inequality:

$$\delta_1^{(s^{n+1}-1)/(s-1)} L_5^{d_{\mathbf{c}} s^n} > L_5^{d_{\mathbf{c}}} \quad \text{for all } n \in \mathbb{N}. \tag{6.8}$$

Indeed, for $n = 1$, (6.8) follows from the choice of L_5 (see inequality (6.7)). Now, assume $n \geqslant 1$ and (6.8) holds for n. Now,

$$\begin{aligned}\delta_1^{(s^{n+2}-1)/(s-1)} L_5^{d_{\mathbf{c}} s^{n+1}} &= \delta_1^{(s^{n+1}-1)/(s-1)}(\delta_1^s L_5^{d_{\mathbf{c}} s})^{s^n}\\ &> \delta_1^{(s^{n+1}-1)/(s-1)}(\delta_1^{s+1} L_5^{d_{\mathbf{c}} s})^{s^n} \quad \text{since } \delta_1 < 1\\ &\geqslant \delta_1^{(s^{n+1}-1)/(s-1)}(L_5^{d_{\mathbf{c}}})^{s^n} \quad \text{by (6.7)}\\ &> L_5^{d_{\mathbf{c}}} \quad \text{by assumption.}\end{aligned}$$

Hence, by induction we complete the proof of the claim.

If $\lambda \in V_{L_5} \setminus \{\eta\}$, then, by Proposition 6.3 and inequality (6.8), we have

$$\begin{aligned}|f_\lambda^n(\mathbf{c}(\lambda))|_v &\geqslant \delta_1^{(s^{n+1}-1)/(s-1)}|z(\lambda)|_v^{d_{\mathbf{c}}s^n}\\ &\geqslant \delta_1^{(s^{n+1}-1)/(s-1)}L_5^{d_{\mathbf{c}}s^n}\\ &\geqslant L_5^{d_{\mathbf{c}}} \geqslant 1,\end{aligned}$$

which means that $M_n(\lambda) = |A_{\mathbf{c},n}(\lambda)|_v$. So,

$$\begin{aligned}M_{n+1}(\lambda) &= |A_{\mathbf{c},n+1}(\lambda)|_v\\ &= |A_{\mathbf{c},n}(\lambda)^{d_P}|_v \cdot \left|P_\lambda\left(1, \frac{B_{\mathbf{c},n}(\lambda)}{A_{\mathbf{c},n}(\lambda)}\right)\right|_v\\ &= M_n(\lambda)^d \cdot \left|P_\lambda\left(1, \frac{1}{f_\lambda^n(\mathbf{c}(\lambda))}\right)\right|_v \quad \text{since } d_P = d\\ &= M_n(\lambda)^d \cdot \left|c_{P,0} + \sum_{i=1}^{d_P} \frac{\mathbf{c}_i(\lambda)}{f_\lambda^n(\mathbf{c}(\lambda))^i}\right|_v\\ &= M_n(\lambda)^d \cdot \left|c_{P,0} + \sum_{i=1}^{d_P} \left(\frac{\mathbf{c}_i(\lambda)}{\mathbf{c}(\lambda)^i}\right)\left(\frac{\mathbf{c}(\lambda)}{f_\lambda^n(\mathbf{c}(\lambda))}\right)^i\right|_v.\end{aligned}$$

Since we have $|f_\lambda^n(\mathbf{c}(\lambda))/\mathbf{c}(\lambda)|_v > 2$ whenever $\lambda \in V_{L_5} \setminus \{\eta\} \subset V_{L_2} \setminus \{\eta\}$ (by (6.6)), we obtain

$$\left|\sum_{i=1}^{d_P} \left(\frac{\mathbf{c}_i(\lambda)}{\mathbf{c}(\lambda)^i}\right)\left(\frac{\mathbf{c}(\lambda)}{f_\lambda^n(\mathbf{c}(\lambda))}\right)^i\right|_v \leqslant \left(\sum_{i=1}^{d_P} \left|\frac{\mathbf{c}_i(\lambda)}{\mathbf{c}(\lambda)^i}\right|_v\right).$$

Because $d_{\mathbf{c}} > \deg(\mathbf{c}_i)/i$, we have

$$\left|\frac{\mathbf{c}_i(\lambda)}{\mathbf{c}(\lambda)^i}\right|_v \to 0 \quad \text{as } \lambda \to \eta \ v\text{-adically};$$

so, there exists $L_4 \geqslant L_5$ such that, for $\lambda \in V_{L_4} \setminus \{\eta\}$, we have

$$\frac{|c_{P,0}|_v}{2} \leqslant \left|P_\lambda\left(1, \frac{B_{\mathbf{c},n}(\lambda)}{A_{\mathbf{c},n}(\lambda)}\right)\right|_v \leqslant \frac{3|c_{P,0}|_v}{2}. \tag{6.9}$$

This concludes the proof of Proposition 6.5. □

7. *Definition of the metrics*

We begin with the following lemma that we will use throughout this section.

LEMMA 7.1. *Let $\mathbf{w} : X \to \mathbb{P}^1$ be a morphism given by $\mathbf{w} := \mathbf{u}/\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in A$ such that $(\mathbf{u}, \mathbf{v}) = A$. Then the line bundle $\mathbf{w}^*\mathcal{O}_{\mathbb{P}^1}(1)$ is linearly equivalent to a multiple of η. Furthermore, if $\deg(\mathbf{u}) > \deg(\mathbf{v})$, then $\mathbf{w}^*\mathcal{O}_{\mathbb{P}^1}(1)$ is linearly equivalent to $\deg(\mathbf{u})\eta$.*

Proof. We have that $\mathbf{w}^*\mathcal{O}_{\mathbb{P}^1}(1)$ equals

$$d_{\mathbf{w}}\eta + \sum_i n_i P_i,$$

where (P_i, n_i) are the zeros P_i with corresponding multiplicities n_i of $\mathbf{v}$. Note $d_{\mathbf{w}} = (\deg(\mathbf{u}) - \deg(\mathbf{v})) > 0$ if and only if the order of the pole of $\mathbf{u}$ at η is larger than the order of the pole of $\mathbf{v}$ at η. On the other hand, $\mathbf{v}$ is itself a map from X to $\mathbb{P}^1$, so $\sum_i n_i P_i$ is linearly equivalent to $\deg(\mathbf{v})\eta$. Thus, $\mathbf{w}^*\mathcal{O}_{\mathbb{P}^1}(1)$ is linearly equivalent with $(d_{\mathbf{w}} + \deg(\mathbf{v}))\eta = \deg(\mathbf{u})\eta$, as desired. □

Now, let $v \in \Omega_K$ be any place of K. We put a family of metrics $\|\cdot\|_{v,n}$ on $\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1)$ for every positive integer n as follows. Since $\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by pull-backs of global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$, it suffices to describe the metric for sections of the form $z = \mathbf{c}^*(u_0t_0 + u_1t_1)$ where t_0 and t_1 are the usual coordinate functions on $\mathbb{P}^1$ and u_0, u_1 are scalars. For a point $\lambda \in Y(\mathbb{C}_v)$, we then define a metric for each $n \in \mathbb{N}$ and each place v as

$$\|z(\lambda)\|_{v,n} := \frac{|u_0\mathbf{a}(\lambda) + u_1\mathbf{b}(\lambda)|_v}{\{\max(|A_{\mathbf{c},n}(\lambda)|_v, |B_{\mathbf{c},n}(\lambda)|_v)\}^{1/d^n}}. \tag{7.1}$$

Furthermore, we define

$$\|z(\eta)\|_{v,n} = v\text{-}\lim_{\lambda\to\eta} \|z(\lambda)\|_{v,n} = \frac{|u_0|_v}{|c_{P,0}^{(d^n-1)/(d^{n+1}-d^n)}|_v}.$$

(Note that the leading coefficient of $A_{\mathbf{c},n}$ is $c_{P,0}^{(d^n-1)/(d-1)} c_{\mathbf{a}}^{d^n}$ according to Proposition 5.1. Recall $\deg(A_{\mathbf{c},n}) = d_{\mathbf{a}}d^n > \deg(B_{\mathbf{c},n})$.)

One arrives at (7.1) as follows. Let $\Phi_{\mathbf{c},n} : X \to \mathbb{P}^1$ be defined by $\Phi_{\mathbf{c},n}(\lambda) = [A_{\mathbf{c},n}(\lambda) : B_{\mathbf{c},n}(\lambda)]$ for $\lambda \neq \eta$ and $\Phi_{\mathbf{c},n}(\eta) = \infty$. Then, since $A_{\mathbf{c},n}$ has a higher order pole at η than $B_{\mathbf{c},n}$, we see that $\Phi_{\mathbf{c},n}$ sends η to $[1:0]$. By Lemma 7.1, we see then that $(\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1))^{d^n}$ is isomorphic to $\Phi_{\mathbf{c},n}^*\mathcal{O}_{\mathbb{P}^1}(1)$. Thus, $\mathbf{c}^*t_0^{\otimes d^n}$ and $\mathbf{c}^*t_1^{\otimes d^n}$ are both sections of $\Phi_{\mathbf{c},n}^*\mathcal{O}_{\mathbb{P}^1}(1)$. Note that $\mathbf{c}^*t_0^{\otimes d^n}$ and $\mathbf{c}^*t_1^{\otimes d^n}$ have no common zero since t_0 and t_1 have no common zero; hence they generate $\Phi_{\mathbf{c},n}^*\mathcal{O}_{\mathbb{P}^1}(1)$ as a line bundle. Likewise, $\Phi_{\mathbf{c},n}^*t_0$ and $\Phi_{\mathbf{c},n}^*t_1$ have no common zero and thus generate $\Phi_{\mathbf{c},n}^*\mathcal{O}_{\mathbb{P}^1}(1)$ as a line bundle. Thus (by [**17**, Section II.6], for example), we have an isomorphism $\tau : (\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1))^{\otimes d^n} \tilde{\to} \Phi_{\mathbf{c},n}^*\mathcal{O}_{\mathbb{P}^1}(1)$, given by $\tau : \mathbf{c}^*t_0^{\otimes d^n} \mapsto \Phi_{\mathbf{c},n}^*t_0$ and $\tau : \mathbf{c}^*t_1^{\otimes d^n} \mapsto \Phi_{\mathbf{c},n}^*t_1$. Now, for each place v, let $\|\cdot\|_v'$ be the metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ given by

$$\|(u_0t_0 + u_1t_1)\|_v'([a:b]) = \frac{|u_0a + u_1b|_v}{\max(|a|_v, |b|_v)}$$

(this is the semipositive metric that gives rise to the usual height on $\mathbb{P}^1$). Then $\|\cdot\|_{v,n}$ is simply the d^nth root of $\tau^*\Phi_{\mathbf{c},n}^*\|\cdot\|_v'$. In particular, the adelic metrized line bundle $\bar{L}_n$ given by $\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1)$ with the metrics $\|\cdot\|_{v,n}$ is isomorphic to a power of the pull-back of a semipositive metrized line bundle, so it is therefore itself semipositive, since the pull-back of a family of metrics satisfying (SP1)–(SP3) will also be a family of metrics satisfying (SP1)–(SP3) (see [**31**, Section 2]).

Remark 7.2. We also note that, for any given model $\mathcal{X}$ for X over the ring of integers $\mathfrak{o}_K$ of K, there exists a finite subset S of places of K depending on $\mathcal{X}, \mathbf{c}$ and $\mathbf{f}$ such that $\Phi_{\mathbf{c},n}$ extends to a morphism from $\mathcal{X}$ to $\mathbb{P}^1$ over the ring of S-integers $\mathfrak{o}_S$ of K for all n. From this, we conclude that the family of metrics $\|\cdot\|_{v,n}$ are the same for all n and all $v \notin S$; this information is used later in our arguments since in our proof we will employ Corollary 4.3.

Indeed, we note that $\Phi_{\mathbf{c},n}$ has good reduction for all nonarchimedean primes that do not divide the (constant) resultant of the family $\mathbf{f}_\lambda$ and also do not divide the leading coefficients of both P and of $\mathbf{a}$. In particular, this proves that, for each such place v of good reduction, for each $\lambda \in Y$ and for each integer n, we have

$$\max\{|A_{\mathbf{c},n}(\lambda)|_v, |B_{\mathbf{c},n}(\lambda)|_v\} = \max\{|\mathbf{a}(\lambda)|_v, |\mathbf{b}(\lambda)|_v\}^{d^n}.$$

Indeed, if $|\mathbf{a}(\lambda)|_v \leqslant |\mathbf{b}(\lambda)|_v$, then dividing equations (5.6) and (5.7) by $|\mathbf{b}(\lambda)|_v^{d^n}$ and using the fact that $\mathbf{c}(\lambda)$ is integral at v while $\Phi_{\mathbf{c},n}$ has good reduction at v we conclude

$$\max\left\{\frac{|A_{\mathbf{c},n}(\lambda)|_v}{|\mathbf{b}(\lambda)|_v^{d^n}}, \frac{|B_{\mathbf{c},n}(\lambda)|_v}{|\mathbf{b}(\lambda)|_v^{d^n}}\right\} = 1.$$

Similarly, if $|\mathbf{a}(\lambda)|_v > |\mathbf{b}(\lambda)|_v$, then $|\mathbf{c}(\lambda)|_v > 1$ and since v is a place of good reduction for $\Phi_{\mathbf{c},n}$ we obtain $|A_{\mathbf{c},n}(\lambda)|_v = |\mathbf{a}(\lambda)|_v^{d^n} > |B_{\mathbf{c},n}(\lambda)|_v$. In conclusion,

$$\max\{|A_{c,n}(\lambda)|_v, |B_{c,n}(\lambda)|_v\} = \max\{|\mathbf{a}(\lambda)|_v, |\mathbf{b}(\lambda)|_v\}^{d^n},$$

as claimed. Now, let S be set of archimedean places along with the nonarchimedean places v which divide the leading coefficient of $\mathbf{a}$ or P or divide the constant resultant of the family $\mathbf{f}_\lambda$. Then

$$\|\cdot\|_{v,n} = \|\cdot\|_{v,0} \quad \text{for all } v \notin S \text{ and all positive integers } n. \tag{7.2}$$

PROPOSITION 7.3. *For any $v \in \Omega_K$, the sequence of metrics $\|\cdot\|_{v,n}$ defined above converges uniformly on $X(\mathbb{C}_v)$.*

Proof. If $\lambda = \eta$, then the convergence is clear. For each $\lambda \in Y(\mathbb{C}_v)$, we define

$$h_{v,n}(\lambda) := \max\{|A_{\mathbf{c},n}(\lambda)|_v, |B_{\mathbf{c},n}(\lambda)|_v\}.$$

(Previously, when v was fixed in our arguments, we denoted $h_{v,n}(\lambda)$ by $M_n(\lambda)$.) Then, for a section of the form $z = \mathbf{c}^*(u_0t_0 + u_1t_1)$ we have

$$\|z(\lambda)\|_{v,n} = \frac{|u_0\mathbf{a}(\lambda) + u_1\mathbf{b}(\lambda)|_v}{h_{v,n}(\lambda)^{1/d^n}}.$$

To show that $\log\|\cdot\|_{v,n}$ converge uniformly, it suffices to show that $\log(h_{v,n}(\lambda)/d^n)$ converge uniformly for $\lambda \in Y(\mathbb{C}_v)$.

Propositions 6.1 and 6.5 show that there exist positive real numbers C_9 and C_{10} such that, for all $n \geqslant 0$, we have

$$C_9 h_{v,n}(\lambda)^d \leqslant h_{v,n+1}(\lambda) \leqslant C_{10} h_{v,n}(\lambda)^d. \tag{7.3}$$

Taking logarithms in (7.3) and dividing by d^{n+1} yields

$$\left|\frac{1}{d^{n+1}}\log h_{v,n+1}(\lambda) - \frac{1}{d^n}\log h_{v,n}(\lambda)\right| \leqslant C_{11}/d^{n+1}$$

for some positive constant C_{11}. Thus, by the usual telescoping series argument, we have

$$\left|\frac{1}{d^m}\log h_{v,m}(\lambda) - \frac{1}{d^n}\log h_{v,n}(\lambda)\right| \leqslant \frac{C_{11}}{d^n}\sum_{i=0}^{\infty}(1/d^i) = \frac{C_{11}}{d^n(1-1/d)}$$

for any $m > n$. Since $C_{11}/d^n(1-1/d)$ can be made arbitrarily small by choosing n large, this gives uniform convergence for all $\lambda \in Y(\mathbb{C}_v)$. □

For each $v \in \Omega_K$, we let $\|\cdot\|_v$ denote the limit of the family of metrics $\|\cdot\|_{v,n}$. Proposition 7.3 shows that the adelic metrized line bundle $\bar{L} = (\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_v\}_{v\in\Omega_K})$ is semipositive. Let $\lambda \in X(\bar{K})$ and choose a meromorphic section s of L whose support is disjoint from the Galois conjugates $\lambda^{[1]}, \dots, \lambda^{[\ell]}$ of λ over K. Furthermore, Lemma 7.1 says that the line bundle $\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1)$ is isomorphic to $L_\eta^{\otimes d_{\mathbf{a}}}$ where L_η is the line bundle determined by the divisor class containing η. As in Section 4, we put

$$h_{\mathbf{c}}(\lambda) := \frac{1}{d_{\mathbf{a}}}\sum_{v\in\Omega_K}\frac{N_v}{\ell}\sum_{i=1}^{\ell} -\log\|s(\lambda^{[i]})\|_v. \tag{7.4}$$

Consequently, $h_{\mathbf{c}} = h_{\bar{L}_\eta}$ is a height function associated to the metrized line bundle $\bar{L}_\eta$ that corresponds to the divisor class containing η. Now we are ready to give a proof of Theorem 5.4.

Proof of Theorem 5.4. Recall that we are given $\mathbf{f}(x) = P(x)/Q(x)$ of degree $d \geqslant 2$ over F and a point $\mathbf{c} \in F$ such that the sequence $\{\deg \mathbf{f}^n(\mathbf{c})\}_{n \geqslant 0}$ is unbounded. Note that $\mathbf{f}$ satisfies the conditions that the resultant $R(\mathbf{f}) \in K^*$ and $d_P \geqslant d_Q + 2$ (equivalently, the point $x = \infty$ is a superattracting fixed point for $\mathbf{f}$).

The first step is to compute the canonical height $\hat{h}_{\mathbf{f}}(\mathbf{c})$ of $\mathbf{c} \in \mathbb{P}^1(F)$ associated to the given morphism $\mathbf{f}$ over $F = K(X)$. We note that F is a product formula field and, moreover, the set of places Ω_F is in one-to-one correspondence with the set of closed points of X over K. Let $e_n = \max\{\deg A_{\mathbf{c},n}, \deg B_{\mathbf{c},n}\}$. Then

$$\begin{aligned}\hat{h}_{\mathbf{f}}(\mathbf{c}) &= \lim_{n\to\infty} \frac{1}{d^n} \sum_{P \in X} \deg(P) \max\{-\operatorname{ord}_P(A_{\mathbf{c},n}), -\operatorname{ord}_P(B_{\mathbf{c},n})\}\\ &= \lim_{n\to\infty} \frac{1}{d^n} \max\{\deg(A_{\mathbf{c},n}), \deg(B_{\mathbf{c},n})\} \quad \text{since } (A_{\mathbf{c},n}, B_{\mathbf{c},n}) = \mathbf{A}\\ &= \lim_{n\to\infty} \frac{e_n}{d^n},\end{aligned}$$

where in the sum P runs over all closed points of X and $\deg(P)$ denotes the degree of P over K. Now we put $g(\lambda) := \hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))/\,\hat{h}_{\mathbf{f}}(\mathbf{c})$, which gives a function on $Y(\bar{K})$. We claim $g(\lambda) = h_{\mathbf{c}}(\lambda)$ for all λ. Since $h_{\mathbf{c}}$ is a height function associated to the divisor class containing η, Theorem 5.4 will follow from the claim.

To prove the claim, we first observe that $g(\lambda)$ is independent of the choice of the point in the orbit $\mathcal{O}_{\mathbf{f}}(\mathbf{c}) = \{\mathbf{f}^n(\mathbf{c})\}_{n \geqslant 0}$. This can be seen as follows: for each $n \geqslant 0$,

$$\frac{\hat{h}_{\mathbf{f}_\lambda}(\mathbf{f}_\lambda^n(\mathbf{c}(\lambda)))}{\hat{h}_{\mathbf{f}}(\mathbf{f}^n(\mathbf{c}))} = \frac{d^n\,\hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda))}{d^n\,\hat{h}_{\mathbf{f}}(\mathbf{c})} = g(\lambda).$$

In the following, we choose n large enough so that $\deg(\mathbf{f}^n(\mathbf{c})) > m$ where $m = m_1 + m_2$, as defined in (5.4). This is possible as $\deg(\mathbf{f}^n(\mathbf{c}))$ is unbounded (actually, in this case $\deg(\mathbf{f}^n(\mathbf{c})) \to \infty$ when $n \to \infty$ since $\mathbf{f}$ is superattracting at ∞). Replacing $\mathbf{c}$ by $\mathbf{f}^n(\mathbf{c})$ if necessary, we may assume that $\mathbf{c}$ satisfies $d_{\mathbf{c}} = \deg(\mathbf{c}) > m$. Then, according to Proposition 5.1 we have $e_n = \max\{\deg A_{\mathbf{c},n}, \deg B_{\mathbf{c},n}\} = d_{\mathbf{a}} d^n$. Hence, $\hat{h}_{\mathbf{f}}(\mathbf{c}) = e_n/d^n = d_{\mathbf{a}}$. Let $\lambda^{[1]}, \ldots, \lambda^{[\ell]}$ be the Galois conjugates of λ over K and let s be a section of $\mathbf{c}^*\mathcal{O}_{\mathbb{P}^1}(1)$ whose support is disjoint from $\lambda^{[1]}, \ldots, \lambda^{[\ell]}$. Now we compute

$$\begin{aligned}d_{\mathbf{a}} h_{\mathbf{c}}(\lambda) &= \sum_{v \in \Omega_K} \frac{N_v}{\ell} \sum_{i=1}^{\ell} -\log \|s(\lambda^{[i]})\|_v\\ &= \sum_{v \in \Omega_K} \frac{N_v}{\ell} \sum_{i=1}^{\ell} \lim_{n\to\infty} \frac{\log\max\{|A_{\mathbf{c},n}(\lambda^{[i]})|_v, |B_{\mathbf{c},n}(\lambda^{[i]})|_v\}}{d^n} - \log|s(\lambda^{[i]})|_v\\ &= \sum_{v \in \Omega_K} \frac{N_v}{\ell} \sum_{i=1}^{\ell} \lim_{n\to\infty} \frac{\log\max\{|A_{\mathbf{c},n}(\lambda^{[i]})|_v, |B_{\mathbf{c},n}(\lambda^{[i]})|_v\}}{d^n} \quad \text{by the product formula,}\\ &= \hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}(\lambda)) \quad \text{see [25, Theorem 5.59]}\\ &= \hat{h}_{\mathbf{f}}(\mathbf{c}) g(\lambda) \quad \text{by the definition of } g(\lambda).\end{aligned}$$

As remarked above, we have $\hat{h}_{\mathbf{f}}(\mathbf{c}) = d_{\mathbf{a}}$. It follows that $g(\lambda) = h_{\mathbf{c}}(\lambda)$ and the proof of Theorem 5.4 is completed. □

8. *Preperiodic points for families of dynamical systems*

We are ready to prove our main results.

Proof of Theorem 2.1. We let K be a number field such that $\mathbf{f}_i$ and also $\mathbf{c}_i$ are defined over K (for $i = 1, 2$). Let $h_{\mathbf{c}_i}(\lambda) := \hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_i)/\hat{h}_{\mathbf{f}}(\mathbf{c})$ be the height function defined as in Section 7 for $i = 1, 2$. As in the proof of Theorem 5.4, $h_{\mathbf{c}_i} = h_{\bar{L}_{\eta,i}}$ is the height function associated to the adelic metrized line bundle $\bar{L}_{\eta,i} = (L_\eta, \{\|\cdot\|_{v,i}\}_{v\in\Omega_K})$ where, for any $v \in \Omega_K$, the metric $\{\|\cdot\|_{v,i}\}$ denotes the limits of the metrics constructed in (7.1) corresponding to $\mathbf{c}_1$ and, respectively, $\mathbf{c}_2$.

Let z be an arbitrary section. Using Remark 7.2 (see (7.2)), we obtain that, for all but finitely many places v, and for all λ such that $|\lambda| > 1$, we have $\|z(\lambda)\|_{v,1} = \|z(\lambda)\|_{v,2}$. Therefore, our hypothesis and Proposition 7.3 allow us to use Corollary 4.3 and hence we conclude the equality of the two metrics; that is,

$$\frac{\hat{h}_{\mathbf{f}_{\lambda,1}}(\mathbf{c}_1(\lambda))}{\hat{h}_{\mathbf{f}_1}(\mathbf{c}_1)} = \frac{\hat{h}_{\mathbf{f}_{\lambda,2}}(\mathbf{c}_1(\lambda))}{\hat{h}_{\mathbf{f}_2}(\mathbf{c}_2)}.$$

So, we have

$$\hat{h}_{\mathbf{f}_{\lambda,1}}(\mathbf{c}_1(\lambda)) = 0 \quad \text{if and only if} \quad \hat{h}_{\mathbf{f}_{\lambda,2}}(\mathbf{c}_1(\lambda)) = 0.$$

This concludes the proof of Theorem 2.1. □

The following results are easy consequences of Theorem 2.1.

COROLLARY 8.1. *Let $\mathbf{c}_i$ and $f_{\lambda,i}$ be as in Theorem 2.1 for $i = 1, 2$. Then, for each $\lambda \in \bar{\mathbb{Q}}$, $\mathbf{c}_1(\lambda)$ is preperiodic for $f_{\lambda,1}$ if and only if $\mathbf{c}_2(\lambda)$ is preperiodic for $f_{\lambda,2}$.*

Proof. Since $\hat{h}_{f_{\lambda,i}}(x) = 0$ if and only if x is preperiodic for $f_{\lambda,i}$ (because $f_{\lambda,i} \in \bar{\mathbb{Q}}(x)$), the conclusion is immediate. □

Proof of Theorem 1.1. We let $\mathbf{c}_1 = a/1$ and $\mathbf{c}_2 = b/1$; by our assumption, $\mathbf{c}_i$ is a quotient of two functions in $\mathbf{A}$ (which generate $\mathbf{A}$), and also $\deg(\mathbf{f}^n(\mathbf{c}_i))$ is unbounded as $n \to \infty$. Since $P_i, Q_i \in \bar{\mathbb{Q}}[x]$ and also $a, b \in \bar{\mathbb{Q}}$, Corollary 5.3 yields that if a (or b) is preperiodic under $\mathbf{f}_\lambda$ (or $\mathbf{g}_\lambda$), then $\lambda \in \bar{\mathbb{Q}}$. Note that $\mathbf{f}_\lambda$, $\mathbf{g}_\lambda$, $\mathbf{c}_1(\lambda)$ and $\mathbf{c}_2(\lambda)$ satisfy the hypothesis of Theorem 2.1. Using Corollary 8.1, we obtain that a is preperiodic under the action of $\mathbf{f}_\lambda$ if and only if b is preperiodic under the action of $\mathbf{g}_\lambda$. □

The following corollary generalizes Theorem 1.1 and its proof is identical with the proof of Theorem 1.1.

COROLLARY 8.2. *Let $P_i, Q_i, R_i \in \bar{\mathbb{Q}}[x]$ be nonzero polynomials such that $\deg(P_i) \geqslant \deg(Q_i) + \deg(R_i) + 2$, and let $a, b \in \bar{\mathbb{Q}}$ such that $Q_1(a)$, $R_1(a)$, $Q_2(b)$ and $R_2(b)$ are all nonzero. Let C be a projective nonsingular curve defined over $\bar{\mathbb{Q}}$, let $\eta \in C(\bar{\mathbb{Q}})$ and let $\mathbf{A}$ be the ring of functions on C regular on $C \setminus \{\eta\}$. Let $\Phi, \Psi \in \mathbf{A}$ be nonconstant functions. If there exist infinitely many $\lambda \in C(\bar{\mathbb{Q}})$ such that both a and b are preperiodic under the action of $\mathbf{f}_\lambda(x) = P_1(x)/Q_1(x) + \Phi(\lambda) \cdot R_1(x)$ and of $\mathbf{g}_\lambda(x) = P_2(x)/Q_2(x) + \Psi(\lambda) \cdot R_2(x)$, respectively, then for all $\lambda \in C(\mathbb{C})$, a is preperiodic for $\mathbf{f}_\lambda$ if and only if b is preperiodic for $\mathbf{g}_\lambda$.*

We have also the following corollary.

COROLLARY 8.3. *Let $P_i, Q_i, R_i \in \bar{\mathbb{Q}}[x]$ be nonconstant polynomials such that $\deg(P_i) > \deg(Q_i) + \deg(R_i)$ for $i = 1, 2$. Let $c_1, c_2 \in \bar{\mathbb{Q}}$ such that c_1 is preperiodic under the action of $P_1(x)/Q_1(x)$, while c_2 is not preperiodic under the action of $P_2(x)/Q_2(x)$. Then, for any two* *nonconstant polynomials* $g_1, g_2 \in \bar{\mathbb{Q}}[x]$ *such that* $g_1(0) = g_2(0) = 0$, *there exist at most finitely many* $\lambda \in \bar{\mathbb{Q}}$ *such that both* $g_1(\lambda) + c_1$ *and* $g_2(\lambda) + c_2$ *are preperiodic under the actions of* $P_1(x)/Q_1(x) + \lambda \cdot R_1(x)$ *and of* $P_2(x)/Q_2(x) + \lambda \cdot R_2(x)$, *respectively.*

Proof. By our assumption on the degrees of P_i, Q_i, R_i, g_i for $i = 1, 2$, we conclude that the conditions in Theorem 2.1 are satisfied for $f_{\lambda,i}(x) := P_i(x)/Q_i(x) + \lambda R_i(x)$, and $\mathbf{c}_i(\lambda) = g_i(\lambda) + c_i$ for $i = 1, 2$. Assume that there exist infinitely many $\lambda \in \bar{\mathbb{Q}}$ such that $g_i(\lambda) + c_i$ is preperiodic under the action of $f_{\lambda,i}(x)$ for $i = 1, 2$. Using Corollary 8.1, we obtain that $g_1(\lambda) + c_1$ is preperiodic under the action of $f_{\lambda,1}$ if and only if $g_2(\lambda) + c_2$ is preperiodic under the action of $f_{\lambda,2}$. However, $c_1 = g_1(0) + c_1$ is preperiodic under $f_{0,1}(x) = P_1(x)/Q_1(x)$, while $c_2 = g_2(0) + c_2$ is not preperiodic under $f_{0,2}(x) = P_2(x)/Q_2(x)$. This contradiction proves that indeed there exist at most finitely many $\lambda \in \bar{\mathbb{Q}}$ such that $g_i(\lambda) + c_i$ is preperiodic under the action of $f_{\lambda,i}(x)$ for $i = 1, 2$. □

Now we prove Theorem 1.3.

Proof of Theorem 1.3. We let $\tilde{C} := C \cup \{\eta\}$ be the projective closure of C in $\mathbb{P}^2$, where η is the point at infinity. We let $\mathbf{A}$ be the ring of functions on $\tilde{C}$ that are regular on C. Then $\mathbf{f}_X(z) := f(z) + X \in \mathbf{A}[z]$ and also $\mathbf{g}_Y(z) := g(z) + Y \in \mathbf{A}[z]$ where X and Y are the corresponding regular functions on C, that is, the functions giving the coordinates of any point. For any critical points c_1 and c_2 of $f(z)$ and of $g(z)$, respectively, we let $\mathbf{c}_1 := c_1/1 \in \mathbf{A}$ and $\mathbf{c}_2 := c_2/1 \in \mathbf{A}$. Then all hypotheses of Theorem 2.1 are satisfied. Therefore, Theorem 2.1 yields that, for each $(x, y) \in C(\mathbb{C})$, we have that c_1 is preperiodic for $\mathbf{f}_x$ if and only if c_2 is preperiodic for $\mathbf{g}_y$ (note that Corollary 5.3 yields that each such (x, y) actually lives over $\bar{\mathbb{Q}}$). Repeating the above analysis for each pair of critical points of f and of g, respectively, we conclude that for each point (x, y) on C, $f(z) + x$ is PCF if and only if $g(z) + y$ is PCF. □

Finally, we can prove Theorem 2.4.

Proof. Recall that we are given the polynomial map $\mathbf{f} \in \mathbf{A}[z]$ of degree d with constant leading coefficients and two points $\mathbf{c}_i = \mathbf{a}_i/\mathbf{b}, i = 1, 2$. In the following, we give a proof under the condition that $\mathbf{b}$ is not a constant since the proof for the case where $\mathbf{b}$ is a constant follows verbatim from [**2**]. Now, since $\mathbf{f}$ has constant leading coefficient, at the expense of replacing K by a finite extension field, $\mathbf{f}$ by a conjugate $\mu^{-1} \circ \mathbf{f} \circ \mu$ and $\mathbf{c}_i$ by $\mu^{-1} \circ \mathbf{c}_i$ (for some suitable linear polynomial μ), we may assume that $\mathbf{f}$ is monic.

The assumption on $\deg(\mathbf{c}_i) = \deg(\mathbf{a}_i) - 2\deg(\mathbf{b}) > M$ guarantees the fact $\deg(\mathbf{f}^n(\mathbf{c}_i)) \to \infty$ as $n \to \infty$ (note that $\mathbf{f}$ is a polynomial, and that M is the largest degree as a function in $\mathbf{A}$ of any coefficient of $\mathbf{f}$). So, we have $B_{\mathbf{c}_i,n} = \mathbf{b}^{d^n}$ for all $n \in \mathbb{N}$. Proposition 5.1 (or by direct computation) shows $\deg(A_{\mathbf{c}_i,n}) = d_{\mathbf{a}_i} d^n$ where $d_{\mathbf{a}_i} := \deg(\mathbf{a}_i)$ for $i = 1, 2$ and the leading coefficients of $A_{\mathbf{c}_i,n}$ are $c_{\mathbf{a}_i}^{d^n}$. It follows that $\mathbf{c}_i, i = 1, 2$ (as well as $\mathbf{f}$) satisfy the conditions in Theorems 2.1 and 5.4. As in the proof of Theorem 5.4, the two points $\mathbf{c}_1$ and $\mathbf{c}_2$ give rise to adelic metrized line bundle $\bar{L}_i = (\mathbf{c}_i^*\mathcal{O}(1), \{\|\cdot\|_{v,i}\}_{v\in\Omega_K})$ such that

$$h_{\bar{L}_i}(\lambda) = \hat{h}_{\mathbf{f}}(\mathbf{c}_i)\, h_{\mathbf{c}_i}(\lambda) = \hat{h}_{\mathbf{f}_\lambda}(\mathbf{c}_i(\lambda)) \quad \text{for all } \lambda \in Y(\bar{K}).$$

It also follows from the proof of Theorem 5.4 that $\hat{h}_{\mathbf{f}}(\mathbf{c}_i) = d_{\mathbf{a}_i}$ and, for $v \in \Omega_K$, the metrics $\|\cdot\|_{v,i}$ are the uniform limits of the following family of metrics for $i = 1, 2$:

$$\|s(\lambda)\|_{v,i,n} := \frac{|\mathbf{b}(\lambda)|_v}{\{\max(|A_{\mathbf{c}_i,n}(\lambda)|_v, |\mathbf{b}(\lambda)|_v^{d^n})\}^{1/d^n}}, \tag{8.1}$$

where $s = \mathbf{c}_1^*(t_1) = \mathbf{c}_2^*(t_1)$. Here we use the fact that $\mathbf{c}_1$ and $\mathbf{c}_2$ have the same denominator $\mathbf{b}$. In the following, for the ease of notation we let $d_i = d_{\mathbf{a}_i}$ for $i = 1, 2$. By Theorem 2.1, we know $h_{\mathbf{c}_1}(\lambda) = h_{\mathbf{c}_2}(\lambda)$ for $\lambda \in Y(\bar{K})$. For $i = 1, 2$, the decomposition of $h_{\mathbf{c}_i}(\lambda)$ as a sum of local heights is given as in (7.4); that is,

$$h_{\mathbf{c}_i}(\lambda) := \frac{1}{d_i} \sum_{v \in \Omega_K} \frac{N_v}{\ell} \sum_{j=1}^{\ell} -\log \|s(\lambda^{[j]})\|_{v,i}, \tag{8.2}$$

where ℓ is the number of Galois conjugates of λ over K. Our hypothesis allows us to apply Theorem 4.2, which says that the two metrics $\|s(\lambda)\|_{v,i}^{1/d_i}, i = 1, 2$ have constant ratio for $\lambda \in Y$ such that $\mathbf{b}(\lambda) \neq 0$. Hence, for each place $v \in \Omega_K$, there exists a constant c_v such that

$$\frac{-\log \|s(\lambda)\|_{v,1}}{d_1} = \frac{-\log \|s(\lambda)\|_{v,2}}{d_2} + c_v.$$

On the other hand, it follows from (8.1) that $-\log \|s(\lambda)\|_{v,i} \geqslant 0$ and the equality of the two heights $h_{\mathbf{c}_1}(\lambda) = h_{\mathbf{c}_2}(\lambda)$, we conclude that $-\log \|s(\lambda)\|_{v,1} = 0$ if and only if $-\log \|s(\lambda)\|_{v,2} = 0$. Hence $c_v = 0$ for each $v \in \Omega_K$. Consequently, we have the equality of the two local heights associated to $\mathbf{c}_1$ and $\mathbf{c}_2$ for each place v. Note

$$-\log \|s(\lambda)\|_{v,i} = \lim_{n\to\infty} \frac{\log^+ |\mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda))|_v}{d^n}.$$

On the other hand, we have $\mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda)) = A_{\mathbf{c}_i,n}(\lambda)/\mathbf{b}(\lambda)^{d^n}$ for all $\lambda \in Y$ such that $\mathbf{b}(\lambda) \neq 0$. By our assumption that $\mathbf{b}$ is not a constant function on X, the set of zeros of $\mathbf{b}$ is not empty and is a subset of $Y(\bar{K}_v)$. Then, the local height function $-\log \|s\|_{v,i}(\lambda)$ has logarithmic singularity around any zero of $\mathbf{b}$. By comparing singularities in Y on both sides of the equality

$$\frac{-\log \|s(\lambda)\|_{v,1}}{d_i} = \frac{-\log \|s(\lambda)\|_{v,2}}{d_2},$$

we conclude $d_1 = d_2 = q$ (for some positive integer q) and hence $d_{\mathbf{c}_1} = d_1 - \deg \mathbf{b} = d_{\mathbf{c}_2}$.

For each place v, there is a sufficiently large L_v such that, for $\lambda \in V_{L_v} \setminus \{\eta\}$, the local heights are $\log^+ |\mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda))|_v = \log |\mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda))|_v$ by Proposition 6.3. Note that the leading coefficients of $A_{\mathbf{c}_i,n}$ is $c_{\mathbf{a}_i}^{d^n}$. Then, for $\lambda \in V_{L_v} \setminus \{\eta\}$ the two local heights associated to $\mathbf{c}_1$ and $\mathbf{c}_2$ have the forms

$$-\log \|s(\lambda)\|_{v,i} = \lim_{n\to\infty} \frac{|A_{\mathbf{c}_i,n}(\lambda)|_v}{d^n} - \log |\mathbf{b}(\lambda)|_v = -d_{\mathbf{c}_i} \log |u(\lambda)|_v + \log |c_{\mathbf{a}_i}|_v + O(|u(\lambda)|_v),$$

where u is a fixed uniformizer of η. As we have the equality of the two local heights $-\log \|s(\lambda)\|_{v,1} = -\log \|s(\lambda)\|_{v,2}$, we conclude $|c_{\mathbf{a}_1}/c_{\mathbf{a}_2}|_v = 1$ for all $v \in \Omega_K$. Therefore, $c_{\mathbf{a}_1} = \zeta c_{\mathbf{a}_2}$ for some root of unity ζ. Note that either $\zeta^{d^r} = 1$ or there exist two positive integers $s > r$ such that $\zeta^{d^s} = \zeta^{d^r}$ and in the latter case we have $(\zeta^{d^r})^{d^{s-r}} = \zeta^{d^r}$. At the expense of replacing $\mathbf{f}$ by an iterate of it, and also replacing $\mathbf{c}_i$ by $\mathbf{f}^r(\mathbf{c}_i)$ (for a suitable $r \in \mathbb{N}$), we may assume $\zeta^d = \zeta$ where d is now the degree of the new map (which is some iterate of $\mathbf{f}$).

We use next the information from an archimedean place v for which we let $|\cdot|_v := |\cdot|$. Noting $A_{\mathbf{c}_i,n}/\mathbf{b}^{d^n} = \mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda))$ and

$$\frac{-\log \|s_i(\lambda)\|_v}{q} = \lim_{n\to\infty} \frac{\log^+ |\mathbf{f}_\lambda^n(\mathbf{c}_i(\lambda))|}{qd^n},$$

we obtain

$$\lim_{n\to\infty} \frac{\log^+ |\mathbf{f}_\lambda^n(\mathbf{c}_1(\lambda))|}{d^n} = \lim_{n\to\infty} \frac{\log^+ |\mathbf{f}_\lambda^n(\mathbf{c}_2(\lambda))|}{d^n} \tag{8.3}$$

for all $\lambda \in Y(\bar{\mathbb{Q}})$. For each λ, there exists a uniformization map φ_λ (using Bötcher's Theorem [8]) satisfying the following properties:

(a) there exists $R_\lambda > 0$ such that $\varphi_\lambda(\mathbf{f}_\lambda(z)) = \varphi_\lambda(z)^d$ for all $z \in \mathbb{C}$ satisfying $|z| > R_\lambda$;
(b) φ_λ is analytic on $\tilde{V}_{R_\lambda} := \{z \in \mathbb{C} : |z| > R_\lambda\}$;
(c) $\varphi_\lambda(z) = z + O(\frac{1}{z})$ and, moreover, φ_λ is univalent on $\tilde{V}_{R_\lambda}$.

More precisely,

$$\varphi_\lambda(z) = z + \sum_{k=1}^{\infty} \frac{b_k(\lambda)}{z^k}, \tag{8.4}$$

where $b_k \in \mathbf{A}$ such that $\deg(b_k) \leqslant M(k+1)$ for each k (the argument is identical as in [2, (5.7)]). Now, if we replace $\mathbf{c}_i$ with $\mathbf{f}^n(\mathbf{c}_i)$ for a sufficiently large positive integer n, then

$$\deg(\mathbf{a}_i) - 2\deg(\mathbf{b}) \geqslant 2M + 1 \qquad \text{for each } i = 1, 2;$$

note that $\deg(\mathbf{f}^n(\mathbf{c}_i)) \to \infty$ as $n \to \infty$. Hence,

$$\sum_{k=1}^{\infty} \frac{b_k(\lambda)}{\mathbf{c}_i(\lambda)^k} = O(|u(\lambda)|^{\deg(\mathbf{b})+1}). \tag{8.5}$$

Letting

$$G_\lambda(z) := \lim_{n\to\infty} \frac{\log^+ |\mathbf{f}_\lambda^n(z)|}{d^n}$$

be the Green's function for the polynomial f_λ, then

$$R_\lambda := \max_{\mathbf{f}'_\lambda(z)=0} e^{G_\lambda(z)}.$$

Properties (a)–(c) of φ_λ allow us to conclude

$$G_\lambda(z) = \log|\varphi_\lambda(z)| \quad \text{if } |z| > R_\lambda.$$

Next, using an identical argument as in [13, Proposition 7.6] (see also [2, Lemma 5.1]) we obtain that, for $L > 0$ sufficiently large, if $\lambda \in V_L \setminus \{\eta\}$, then $\mathbf{c}_i(\lambda) \in \tilde{V}_{R_\lambda}$. Using also (8.3), we conclude that

$$\log|\varphi_\lambda(\mathbf{c}_1(\lambda))| = \log|\varphi_\lambda(\mathbf{c}_2(\lambda))| \tag{8.6}$$

for all $\lambda \in V_L(\bar{\mathbb{Q}}) \setminus \{\eta\}$. Using the fact $\bar{\mathbb{Q}}$ is dense in $\mathbb{C}$, we can extend (8.6) to all $\lambda \in V_L(\mathbb{C}) \setminus \{\eta\}$. Using the Open Mapping Theorem (see [10, Chapter IX, Section 6]) for the meromorphic function

$$\lambda \longmapsto \frac{\varphi_\lambda(\mathbf{c}_1(\lambda))}{\varphi_\lambda(\mathbf{c}_2(\lambda))}$$

defined on the one-dimensional $\mathbb{C}$-manifold X, we conclude that this function must be constant (since it takes values of absolute value equal to 1 for all λ in a neighborhood of η). Hence there exists some $u \in \mathbb{C}$ of absolute value equal to 1 such that

$$\varphi_\lambda(\mathbf{c}_1(\lambda)) = u \cdot \varphi_\lambda(\mathbf{c}_2(\lambda)). \tag{8.7}$$

Actually, using the fact $\deg(\mathbf{c}_1) = \deg(\mathbf{c}_2)$, we conclude $u = \zeta$. Furthermore, equating in (8.7) the meromorphic parts that are of order $|u(\lambda)|^k$ for $k \leqslant \deg(\mathbf{b})$, and also using (8.4) and (8.5), we conclude that $\mathbf{a}_1 = \zeta\mathbf{a}_2$ and so,

$$\mathbf{c}_1 = \zeta\mathbf{c}_2.$$

Moreover, repeating the above argument for each $n \geqslant 1$, we also obtain

$$\mathbf{f}_\lambda^n(\mathbf{c}_1(\lambda)) = \zeta\mathbf{f}_\lambda^n(\mathbf{c}_2(\lambda)) \quad \text{for all } n.$$

Therefore, $z \mapsto \zeta \cdot z$ is an automorphism for the family of polynomials $\mathbf{f}_\lambda$ (see also [2, Section 5.7]) and thus we obtain the conclusion of Theorem 2.4 with the constant family of polynomials $\mathbf{h}(z) = \zeta z$. $\square$

9. *Higher-dimensional case*

In this section, we prove Theorem 1.4; we continue with the notation from Theorem 1.4. By abuse of notation, we denote by $P(x)$ the polynomial $P(X/Z, 1)$ for the variable $x = X/Z$. Similarly, we denote by $Q(y)$ the polynomial $Q(Y/Z, 1)$ with the variable $y = Y/Z$.

Let $a, b \in \bar{\mathbb{Q}}^*$. For each $n \geqslant 0$, we let $A_n(\lambda,\mu), B_n(\lambda,\mu) \in \bar{\mathbb{Q}}[\lambda,\mu]$ such that

$$\mathbf{f}^n_{\lambda,\mu}([a : b : 1]) = [A_n(\lambda,\mu) : B_n(\lambda,\mu) : 1].$$

More precisely, $A_0 = a$ and $B_0 = b$, while, for each $n \geqslant 0$, we have

$$A_{n+1}(\lambda,\mu) = P(A_n(\lambda,\mu)) + \lambda B_n(\lambda,\mu)$$

and

$$B_{n+1}(\lambda,\mu) = Q(B_n(\lambda,\mu)) + \mu A_n(\lambda,\mu).$$

It is easy to check $\deg(A_n) = \deg(B_n) = d^{n-1}$ for all $n \geqslant 1$ (here we use the fact $d \geqslant 3$). In order to apply our method, we consider the following family of metrics corresponding to any section $s := u_0t_0 + u_1t_1 + u_2t_2$ (with scalars u_i) of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ of $\mathbb{P}^2$. Using the coordinates $\lambda = t_0/t_2$ and $\mu = t_1/t_2$ on the affine subset of $\mathbb{P}^2$ corresponding to $t_2 \neq 0$, for each $n \in \mathbb{N}$ we obtain that the metrics $\| \cdot \|_{v,n} := \|s^{(a,b)}\|_{v,n}$ are defined as follows: $\|s([t_0 : t_1 : t_2])\|_{v,n}$ equals

$$\begin{cases} \dfrac{|u_0t_0 + u_1t_1|_v}{\sqrt[d^{n-1}]{\max\{|c_P|_v^{(d^n-1)/(d-1)}|bt_0|_v^{d^{n-1}}, |c_Q|_v^{(d^n-1)/(d-1)}|at_1|_v^{d^{n-1}}\}}} & \text{if } t_2 = 0, \\ \dfrac{|u_0\lambda + u_1\mu + u_2|_v}{\sqrt[d^{n-1}]{\max\{|A_n(\lambda,\mu)|_v, |B_n(\lambda,\mu)|_v, 1\}}} & \text{if } [t_0 : t_1 : t_2] = [\lambda : \mu : 1], \end{cases}$$

where $P(X, 0) = c_PX^d$ and $Q(Y, 0) = c_QY^d$ (with nonzero constants c_P and c_Q as assumed in Theorem 1.4). We note that if we let

$$\tilde{A}_n(t_0, t_1, t_2) := t_2^{d^{n-1}} \cdot A_n\left(\frac{t_0}{t_2}, \frac{t_1}{t_2}\right)$$

and

$$\tilde{B}_n(t_0, t_1, t_2) = t_2^{d^{n-1}} \cdot B_n\left(\frac{t_0}{t_2}, \frac{t_1}{t_2}\right),$$

then the map

$$\theta_n : [t_0 : t_1 : t_2] \to [\tilde{A}_n(t_0, t_1, t_2) : \tilde{B}_n(t_0, t_1, t_2) : t_2^{d^{n-1}}]$$

is an endomorphism of $\mathbb{P}^2$. Indeed, if $t_2 = 0$, then we have

$$\tilde{A}_n(t_0, t_1, 0) = c_P^{(d^n-1)/(d-1)}b^{d^{n-1}} \cdot t_0^{d^{n-1}}$$

and

$$\tilde{B}_n(t_0, t_1, 0) = c_Q^{(d^n-1)/(d-1)}a^{d^{n-1}} \cdot t_1^{d^{n-1}},$$

which ensures that the above map is well defined on $\mathbb{P}^2$ (note that a and b are nonzero). Thus, we have an isomorphism $\tau : \mathcal{O}_{\mathbb{P}^2}(d^{n-1}) \tilde{\to} \theta_n^*\mathcal{O}_{\mathbb{P}^2}(1)$, given by $\tau : t_0^{d^{n-1}} \mapsto \tilde{A}_n(t_0, t_1, t_2)$, $\tau : t_1^{d^{n-1}} \mapsto \tilde{B}_n(t_0, t_1, t_2)$ and $\tau : t_2^{d^{n-1}} \mapsto t_2^{d^{n-1}}$.

Let $\|\cdot\|_v'$ be the metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ given by

$$\|(u_0t_0+u_1t_1+u_2t_2)\|_v'([a:b:c]) = \frac{|u_0a+u_1b+u_2c|_v}{\max(|a|_v,|b|_v,|c|_v)}.$$

We see then that $\|\cdot\|_{v,n}$ is simply the d^{n-1}th root of $\tau^*\theta_n^*\|\cdot\|_v'$. Note that the degree of θ_n is the same as the total degree of the polynomials A_n and B_n, and thus $\deg\theta_n = d^{n-1}$. Hence, for each n, we have that $\|\cdot\|_{v,n}$ is a semipositive metric on $L=\mathcal{O}_{\mathbb{P}^2}(1)$. We let $\bar{L}_n$ denote the adelic metrized line bundle coming from $\|\cdot\|_{v,n}$.

In order to use Corollary 4.3, we need only prove that the sequence $\{\log\|\cdot\|_{v,n}\}_n$ converges uniformly on $\mathbb{P}^2$ to a metric $\|\cdot\|_v$ on the adelic metrized line bundle $\bar{L}$. Then we would obtain that the height of each point $[\lambda:\mu:1]$ with respect to $\bar{L}$ is

$$h_{\bar{L}}([\lambda:\mu:1]) = d\cdot\hat{h}_{\mathbf{f}_{\lambda,\mu}}([a:b:1]), \tag{9.1}$$

since $h_{\bar{L}_n}([\lambda:\mu:1]) = h(f^n_{\lambda,\mu}([a:b:1]))/d^{n-1}$ and

$$\hat{h}_{\mathbf{f}_{\lambda,\mu}}([a:b:1]) = \lim_{n\to\infty}\frac{h(f^n_{\lambda,\mu}([a:b:1]))}{d^n} = d\lim_{n\to\infty}\frac{h(f^n_{\lambda,\mu}([a:b:1]))}{d^{n-1}}.$$

Clearly, $\{\log\|\cdot\|_{v,n}\}_n$ converges uniformly on the line at infinity from $\mathbb{P}^2$. Indeed, one splits the analysis whether $t_1=0$, or not. In the first case, the convergence is clear, while in the latter case, one further splits the argument into two cases: whether $|t_0/t_1|_v$ is small (say, bounded by some positive real number L), or if $|t_0/t_1|_v > L$. In both cases, the convergence is uniform; we omit the details since they are very similar to (but much simpler than) the analysis that we will do next at points away from the line at infinity.

As before (see Propositions 6.1, 6.5 and 7.3), we will achieve our goal once we prove

$$\frac{M_{n+1}(\lambda,\mu)}{M_n(\lambda,\mu)^d}\quad\text{is uniformly bounded above and below,}$$

where $M_n(\lambda,\mu) := \max\{|A_n(\lambda,\mu)|_v, |B_n(\lambda,\mu)|_v, 1\}$.

Let K be a number field containing a, b and all coefficients of both P and Q. Let $v\in\Omega_K$ be a fixed place. We first observe that there exist real numbers $L_6>1$ and $\delta>0$ (depending only on v and on the coefficients of P and Q) such that, for each $z\in\mathbb{C}_v$ satisfying $|z|_v\geqslant L_6$, we have

$$\min\{|P(z)|_v, |Q(z)|_v\}\geqslant \delta|z|_v^d. \tag{9.2}$$

(Here we use the fact that $\deg(P)=\deg(Q)=d$, which is equivalent to the fact that both $P(X,0)$ and $Q(Y,0)$ are nonzero polynomials.)

Furthermore, there exists a constant $C_{15}>1$ (depending only on v and on the coefficients of P and Q) such that, for each $z\in\mathbb{C}_v$, we have

$$\max\{|P(z)|_v,|Q(z)|_v\}\leqslant C_{15}\cdot\max\{1,|z|_v\}^d. \tag{9.3}$$

Lemma 9.1. *Let L be any real number larger than* 1. *There exist positive real numbers C_{16} and C_{17} depending on v, L and on the coefficients of P and Q such that*

$$C_{16}\leqslant\frac{M_{n+1}(\lambda,\mu)}{M_n(\lambda,\mu)^d}\leqslant C_{17}$$

for all $n\geqslant 1$ and for all $\lambda,\mu\in\mathbb{C}_v$ such that $\max\{|\lambda|_v,|\mu|_v\}\leqslant L$.

To ease the notation, we let $M_n = M_n(\lambda,\mu)$ below if there is no danger of confusion.

Proof. Clearly, by its construction, $M_n\geqslant 1$. Therefore, using (9.3), we get

$$|A_{n+1}(\lambda,\mu)|_v\leqslant|P(A_n(\lambda,\mu))|_v+|\lambda|_v\cdot|B_n(\lambda,\mu)|_v\leqslant C_{15}M_n^d+LM_n\leqslant M_n^d(C_{15}+L)$$

and, similarly,

$$|B_{n+1}(\lambda,\mu)|_v \leqslant |Q(B_n(\lambda,\mu))|_v + |\mu|_v \cdot |A_n(\lambda,\mu)|_v \leqslant M_n^d \cdot (C_{15}+L).$$

This proves the existence of the upper bound C_{17} as in the conclusion of Lemma 9.1.

For the proof of the existence of the lower bound C_{16}, we let $L_7 \geqslant L_6$ be a real number satisfying

$$L_7^{d-1} > \frac{2L}{\delta}. \tag{9.4}$$

Now we split our analysis into two cases.

Case 1. $M_n \leqslant L_7$
In this case, clearly, $M_{n+1}/M_n^d \geqslant 1/L_7^d$.

Case 2. $M_n > L_7$
In this case, without loss of generality, we may assume $|A_n(\lambda,\mu)|_v = M_n$. Then

$$\begin{aligned}
|A_{n+1}(\lambda,\mu)|_v &\geqslant |P(A_n(\lambda,\mu))|_v - |\lambda|_v \cdot |B_n(\lambda,\mu)|_v \\
&\geqslant \delta M_n^d - LM_n \quad \text{ using (9.2) and that } |A_n(\lambda,\mu)|_v > L_7 \geqslant L_6 \\
&\geqslant \delta M_n^d \cdot \left(1 - \frac{L}{\delta M_n^{d-1}}\right) \\
&\geqslant \frac{\delta M_n^d}{2} \quad \text{ using (9.4) and that } M_n > L_7.
\end{aligned}$$

This concludes the proof of Lemma 9.1. □

Now let L be a real number larger than

$$\max\left\{1, \frac{2|Q(b)|_v}{|a|_v}, \frac{2|P(a)|_v}{|b|_v}, \frac{2L_6}{\min\{|b|_v,|a|_v\}}, \frac{\delta L_6^{d-1}}{2}, \frac{2^d}{\delta |a|_v^{d-1}}, \frac{2^d}{\delta |b|_v^{d-1}}\right\}.$$

(Here we use the fact that both a and b are nonzero.)

LEMMA 9.2. *If either $|\lambda|_v > L$, or $|\mu|_v > L$, then*

$$\frac{\delta}{2} \leqslant \frac{M_{n+1}}{M_n^d} \leqslant C_{15} + \frac{\delta}{2}$$

for each $n \geqslant 1$.

Proof. Without loss of generality, we may assume $|\lambda|_v \geqslant |\mu|_v$; hence $|\lambda|_v > L$. We note

$$A_1(\lambda,\mu) = P(a) + b\lambda \quad \text{and} \quad B_1(\lambda,\mu) = Q(b) + a\mu.$$

Then

$$\begin{aligned}
|A_1(\lambda,\mu)|_v &\geqslant |b|_v|\lambda|_v - |P(a)|_v \\
&\geqslant \frac{|b|_v|\lambda|_v}{2} \quad \text{ since } |\lambda|_v > L > \frac{2|P(a)|_v}{|b|_v} \\
&\geqslant \frac{L\cdot|b|_v}{2} \\
&> L_6 \quad \text{ since } L > \frac{2L_6}{|b|_v}.
\end{aligned}$$

CLAIM 9.3. For all $n \geqslant 1$, we have $M_n^{d-1} \geqslant 2|\lambda|_v/\delta$.

Proof. The claim follows by induction on n. In the case $n = 1$, we have

$$\begin{aligned} M_1^{d-1} &\geqslant |A_1(\lambda,\mu)|_v^{d-1} \\ &\geqslant \left(\frac{|b|_v|\lambda|_v}{2}\right)^{d-1} \\ &\geqslant \frac{|b|_v|\lambda|_v}{2}\cdot\left(\frac{|b|_v\cdot L}{2}\right)^{d-2} \quad \text{since } |\lambda|_v > L \\ &\geqslant \frac{2|\lambda|_v}{\delta}\cdot\frac{L}{2^d/\delta|b|_v^{d-1}} \quad \text{since } L>1 \text{ and } d-2\geqslant 1 \\ &\geqslant \frac{2|\lambda|_v}{\delta} \quad \text{since } L > \frac{2^d}{\delta|b|_v^{d-1}}. \end{aligned}$$

Now, assume $M_n^{d-1} \geqslant 2|\lambda|_v/\delta$. First we note that since

$$|\lambda|_v > L > \frac{\delta L_6^{d-1}}{2},$$

we obtain that $M_n > L_6 > 1$. So, if $|A_n(\lambda,\mu)|_v = M_n$, then

$$\begin{aligned} M_{n+1} &= |A_n(\lambda,\mu)|_v \\ &\geqslant |P(A_n(\lambda,\mu))|_v - |\lambda|_v\cdot|B_n(\lambda,\mu)|_v \\ &\geqslant \delta M_n^d - |\lambda|_v M_n \quad \text{using (9.2)} \\ &\geqslant \delta M_n^d\left(1-\frac{|\lambda|_v}{\delta M_n^{d-1}}\right) \\ &\geqslant \frac{\delta}{2}\cdot M_n^d \quad \text{using the inductive hypothesis.} \end{aligned}$$

Similarly, if $|B_n(\lambda,\mu)|_v = M_n$, then

$$\begin{aligned} M_{n+1} &= |B_n(\lambda,\mu)|_v \\ &\geqslant |Q(B_n(\lambda,\mu))|_v - |\mu|_v\cdot|A_n(\lambda,\mu)|_v \\ &\geqslant \delta M_n^d - |\lambda|_v M_n \quad \text{using (9.2) and that } |\lambda|_v \geqslant |\mu|_v \\ &\geqslant \delta M_n^d\left(1-\frac{|\lambda|_v}{\delta M_n^{d-1}}\right) \\ &\geqslant \frac{\delta}{2}\cdot M_n^d \quad \text{using the inductive hypothesis.} \end{aligned}$$

So, the above inequalities yield

$$M_{n+1} \geqslant M_n\cdot\frac{\delta M_n^{d-1}}{2} \geqslant M_n\cdot|\lambda|_v \geqslant M_n\cdot L \geqslant M_n,$$

and thus $M_{n+1}^{d-1} \geqslant M_n^{d-1} \geqslant 2|\lambda|_v/\delta$ as well. □

Furthermore, the above proof actually shows the left-hand side of the inequality from the conclusion of our Lemma 9.2, that is,

$$M_{n+1} \geqslant \frac{\delta}{2}\cdot M_n^d.$$

We are left to prove the right-hand side of the inequality from our conclusion. For this, we use again Claim 9.3, and infer

$$\begin{aligned} M_{n+1} &\leqslant \max\{|P(A_n(\lambda,\mu))|_v, |Q(B(\lambda,\mu))|_v\} + \max\{|\lambda|_v, |\mu|_v\} \cdot M_n(\lambda,\mu) \\ &\leqslant C_{15} M_n^d + |\lambda|_v \cdot M_n \quad \text{using (9.3)} \\ &\leqslant M_n^d \cdot \left(C_{15} + \frac{|\lambda|_v}{M_n^{d-1}} \right) \\ &\leqslant \left(C_{15} + \frac{\delta}{2} \right) \cdot M_n^d, \end{aligned}$$

as desired. □

Therefore (using Lemmas 9.1 and 9.2), arguing precisely as in the proof of Proposition 7.3, we obtain that $\{\log \|s\|_{v,n}\}_n$ converges uniformly on $\mathbb{P}^2$. Indeed, by Lemmas 9.1 and 9.2, we obtain that there exists a positive constant $C_{18} > 1$ such that

$$\frac{1}{C_{18}} \leqslant \frac{M_{n+1}}{M_n^d} \leqslant C_{18}.$$

Hence, there exists a positive constant C_{19} such that, for each $m, n \in \mathbb{N}$ with $n > m$,

$$|\log \|s([\lambda : \mu : 1])\|_{v,n} - \log \|s([\lambda : \mu : 1])\|_{v,m}| \leqslant \frac{C_{19}}{d^m}.$$

Hence the sequence of metrics converges uniformly on $\mathbb{P}^2$. This allows us to use Corollary 4.3 and conclude that, for any given $a_i, b_i \in \bar{\mathbb{Q}}^*$ (for $i = 1, 2$), if there exists a set of points $[\lambda : \mu : 1]$ which is Zariski-dense in $\mathbb{P}^2$ such that, for each such pairs (λ, μ) both $[a_1 : b_1 : 1]$ and $[a_2 : b_2 : 1]$ are preperiodic for $\mathbf{f}_{\lambda,\mu}$, then the two sequences of metrics $\|s^{(a_i,b_i)}\|_{v,n}$ (corresponding to the two starting points $[a_i : b_i : 1]$ for $i = 1, 2$) converge to the *same* metric. Hence, using (9.1), we obtain the equality of the two canonical heights:

$$\hat{h}_{\mathbf{f}_{\lambda,\mu}}([a_1 : b_1 : 1]) = \hat{h}_{\mathbf{f}_{\lambda,\mu}}([a_2 : b_2 : 1]).$$

Therefore, for *each* $(\lambda, \mu) \in \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$, $[a_1 : b_1 : 1]$ is preperiodic for $\mathbf{f}_{\lambda,\mu}$ if and only if $[a_2 : b_2 : 1]$ is preperiodic for $\mathbf{f}_{\lambda,\mu}$. This concludes the proof of Theorem 1.4.

Remark 9.4. If one considers a two-parameter family of endomorphisms $\mathbf{f}_{\lambda,\mu}$ of $\mathbb{P}^1$, then for any two starting points $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{P}^1$, one expects that there exists a Zariski-dense set of parameters (λ, μ) such that both $\mathbf{c}_1$ and $\mathbf{c}_2$ are preperiodic for $\mathbf{f}_{\lambda,\mu}$. Indeed, for each $i = 1, 2$ and for each distinct positive integers m and n there exists a curve $C_{i,m,n}$ in the moduli containing all (λ, μ) such that $\mathbf{f}_{\lambda,\mu}^m(\mathbf{c}_i) = \mathbf{f}_{\lambda,\mu}^n(\mathbf{c}_i)$. Thus generically $C_{1,m,n} \cap C_{2,k,\ell} \neq \emptyset$ (for any two pairs of distinct positive integers (m, n) and (k, ℓ)). Therefore, one would expect

$$\bigcup_{\substack{k,\ell,m,n \in \mathbb{N} \\ k \neq \ell \\ m \neq n}} C_{1,m,n} \cap C_{2,k,\ell}$$

is Zariski-dense in the moduli. Hence the first interesting case when one expects the principle of unlikely intersections in algebraic dynamics holds for a two-dimensional moduli is for a family of endomorphisms of $\mathbb{P}^2$ (as proved in Theorem 1.4).

Remark 9.5. In the case of a family $\mathbf{f}_{\lambda,\mu}$ of endomorphisms of $\mathbb{P}^2$, the right question is indeed whether there exists a Zariski-dense set of points in the moduli for which both starting points $\mathbf{c}_1$ and $\mathbf{c}_2$ are preperiodic. There are examples when there are infinitely many pairs (λ, μ) such that both $\mathbf{c}_1$ and $\mathbf{c}_2$ are preperiodic under $\mathbf{f}_{\lambda,\mu}$, but it is not true that $\mathbf{c}_1$ is preperiodic under $\mathbf{f}_{\lambda,\mu}$ if and only if $\mathbf{c}_2$ is preperiodic under $\mathbf{f}_{\lambda,\mu}$; this happens when the corresponding points $[\lambda : \mu : 1]$ are not Zariski-dense in the moduli $\mathbb{P}^2$. For example, let

$$\mathbf{f}_{\lambda,\mu}([X : Y : Z]) = [X^3 - XZ^2 + \lambda YZ^2 : Y^3 + \mu XZ^2 : Z^3]$$

and $\mathbf{c}_1 = [0 : 1 : 1]$, $\mathbf{c}_2 = [1 : 2 : 1]$. Then $\mathbf{c}_1$ is a fixed point for

$$\mathbf{f}_{0,0}([X : Y : Z]) = [X^3 - XZ^2 : Y^3 : Z^3],$$

while $\mathbf{c}_2$ is not preperiodic for the same map. On the other hand, there exist infinitely many $(\lambda, \mu) \in \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$ such that both $\mathbf{c}_1$ and $\mathbf{c}_2$ are preperiodic for $\mathbf{f}_{\lambda,\mu}$, but they all lie on a line in the moduli.

Indeed, let $\mu = \zeta - 8$ for any root of unity ζ. Then both $\mathbf{c}_1$ and $\mathbf{c}_2$ are preperiodic under the action of

$$\mathbf{f}_{0,\mu}([X : Y : Z]) = [X^3 - XZ^2 : Y^3 + (\zeta - 8)XZ^2 : Z^3].$$

Clearly, $\mathbf{c}_1$ is fixed by any map $\mathbf{f}_{0,\mu}$. On the other hand, $\mathbf{f}_{0,\zeta-8}(\mathbf{c}_2) = [0 : \zeta : 1]$, which is preperiodic under any map $\mathbf{f}_{0,\mu}$ since ζ is a root of unity and $\mathbf{f}^n_{0,\mu}(\mathbf{c}_2) = [0 : \zeta^{3^{n-1}} : 1]$ for any positive integer n.

Remark 9.6. Combining the observations from Remarks 9.4 and 9.5, we also note that, for any positive integers m and n not both equal to 1, if $\mathbf{f}_{\lambda,\mu} : \mathbb{P}^m \to \mathbb{P}^m$ and $\mathbf{g}_{\lambda,\mu} : \mathbb{P}^n \to \mathbb{P}^n$ are two-parameter families of endomorphisms (defined over $\bar{\mathbb{Q}}(\lambda, \mu)$), then one might hope that, given any two starting points $\mathbf{a} \in \mathbb{P}^m_{\lambda,\mu}$ and $\mathbf{b} \in \mathbb{P}^n_{\lambda,\mu}$, if there exists a Zariski-dense set of parameter points $(\lambda, \mu) \in \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$ in the (affine) plane such that both $\mathbf{a}(\lambda, \mu) \in \mathbb{P}^m$ and $\mathbf{b}(\lambda, \mu) \in \mathbb{P}^n$ are preperiodic for $\mathbf{f}_{\lambda,\mu} : \mathbb{P}^m \to \mathbb{P}^m$ and for $\mathbf{g}_{\lambda,\mu} : \mathbb{P}^n \to \mathbb{P}^n$, respectively, then at least one of the following three possibilities must occur:

(1) $\mathbf{a}(\lambda, \mu)$ is preperiodic for $\mathbf{f}_{\lambda,\mu}$ for *all* $\lambda, \mu \in \bar{\mathbb{Q}}$;
(2) $\mathbf{b}(\lambda, \mu)$ is preperiodic for $\mathbf{g}_{\lambda,\mu}$ for *all* $\lambda, \mu \in \bar{\mathbb{Q}}$;
(3) for *each* $(\lambda, \mu) \in \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$, we have that $\mathbf{a}(\lambda, \mu)$ is preperiodic for $\mathbf{f}_{\lambda,\mu}$ *if and only if* $\mathbf{b}(\lambda, \mu)$ is preperiodic for $\mathbf{g}_{\lambda,\mu}$.

However, at the moment the above trichotomy remains still at the speculative level, since the known methods cannot deliver even the simplest case $m = 1$ and $n = 2$. Indeed, in the case where $m = 1$ or $n = 1$, it is not clear how to define the relevant height functions, and thus we do not know if the methods of this paper are applicable at all. When $m, n \geqslant 2$, on the other hand, we do not see an obvious obstacle to applying the methods of this paper, though, of course, we expect very difficult technical complications arising from proving that the corresponding metrics converge uniformly in order to apply the equidistribution statements in arithmetic dynamics (such as Corollary 4.3).

Our Theorem 1.4 is, to our knowledge, the first known positive result for the above trichotomy.

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