1 Introduction

Here is the basic idea we will consider:

You have a space X. Usually an algebraic or analytic space. You have an event (discrete and not too crazy) which takes place infinitely often. This somehow forces X to be rigid.

Example 1.1 $f \in \mathbb{Q}[x, y]$ *such that*

$$f(x, y) = x^{2012} - 203x^5y^7 + x^2y^9 - 21$$

The event is $\exists (x, y) \in \mathbb{Q} \times \mathbb{Q}$ such that $f(x, y) = 0$.

Of course, we have

$$\mathcal{C} := \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}.$$

So, $\mathcal{C}$ is the space and the event is having many $\mathbb{Q}$-rational points.

Theorem 1.2 *(Faltings, Mordell conjecture) When f is* generic *and of degree at least 4, then there are at most finitely many $(x, y) \in \mathbb{Q}^2$ such that $f(x, y) = 0$.*

If $\mathcal{C}$ contains infinitely many $\mathbb{Q}$-rational points, then there are essentially two cases:

- Something like: $x^2 + y^2 = 1$ so $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$ ie, the curve is parameterized.
- Elliptic curves - the points are not parameterized in the same manner as example 1, but from one $\mathbb{Q}$-rational point, one can construct many via the group law.

In each case, there are not only infinitely many $\mathbb{Q}$-rational points, but there is something special about the distribution of $\mathbb{Q}$-rational points is quite special.

Example 1.3 *Now, consider the same space:*

$$\mathcal{C} := \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}.$$

This time, we will change the event to: $\exists (x, y)$ which are roots of unity such that $f(x, y) = 0$.

This event occurs infinitely often when $f(x, y) = cx^m - y^n$ and c is a root of unity. For such curves, for any point $(x, y) \in \mathcal{C}$, x is a root of unity iff y is a root of unity (this is in contrast to the elliptic curve example).

Example 1.4 *Let $f \in \mathbb{Q}[x]$ and start with the point $a = 0$.*

Specifically take $f(x) = x^2 + 1$. So,

$$0 \mapsto 1 \mapsto 2 \mapsto 5 \mapsto 26...$$

Or perhaps let $f(x) = x^2$ and $a = i$. Then,

$$i \mapsto -1 \mapsto 1 \mapsto 1 \mapsto 1..$$

In this case, i is preperiodic *for f.*

Notation: In dynamics, $f^2(a) := f(f(a))$.

Assume you know that infinitely many of the values $f^n(a)$. Can you recover $f(x)$? Imagine that you only know the set of iterates (not which one is the third iterate or the nineth, etc. Then, can you still essentially determine the polynomial f?

Example 1.5 *Assume you are given infinitely many point which are preperiodic for the polynomial $f(x)$. Does this give you some information about $f(x)$? You can pretty much recover $f(x)$ (modulo iterates...).*

One should note that this is not a vacuous question: $f(x)$ certainly has infintely many preperiodic points.