Math 430 Take-Home Midterm Due November 3

1. Let A be a Dedekind domain in a field K, and let L and L' be finite extensions of K, with B and B' the integral closures of A in L and L' respectively. Suppose there is a prime $\mathfrak{p}$ of A such that $\mathfrak{p}B = \mathfrak{q}^{[L:K]}$ (that is, $\mathfrak{p}$ ramifies completely in B) and $\mathfrak{p}$ does not ramify in B'. Show that $L \cap L' = K$. [Hint: How would $\mathfrak{p}$ factor in the integral closure of A in $L \cap L'$?]

2. In all of the following, L is a finite, totally inseparable extension of K of degree p^n, and v is a discrete valuation on K (that is a homomorphism from K^* onto $\mathbb{Z}$ such that $v(x+y) \geq \min(v(x), v(y))$ for all $x, y, x+y \in K^*$).

(a) Define $w : L^* \longrightarrow \mathbb{Z}$ by $w(z) = v(z^{p^n})$ for all $w \in L^*$. Show that there is a positive integer e such that $w' = w/e$ is a discrete valuation on L^*. [Hint: The only reason for the e is that you need to divide by the smallest positive integer value of $v(z^{p^n})$ for $z \in L^*$ in order to make the map surjective onto $\mathbb{Z}$.]

(b) Show that if A is a discrete valuation ring in K, then its integral closure in B is also a discrete valuation ring. [Hint: Use (a).]

3. Let A be an integral domain such that:

(a) $A_{\mathfrak{m}}$ is Noetherian for every maximal ideal $\mathfrak{m}$ of A; and

(b) every nonzero proper ideal of A is contained in finitely many maximal ideals of A.

Show that A is Noetherian. [Hint: I gave a sketch in class and I will repeat it here. Let I be an ideal of A and let $a \in I$ be nonzero. Then a is contained in finitely many maximal ideals $\mathfrak{m}$. For each such $\mathfrak{m}$, there is a finite set of generators in I for the ideal $A_{\mathfrak{m}}I$. Take the ideal J generated by the union of these along with a and show it must equal I (you might find Exercise 2 on Page 17 of the book – which also appeared as a homework problem – useful here).]

4. Let A be a Dedekind domain with field of fractions K and let B be the integral closure of A in a finite, purely inseparable extension of K.

(a) Show that for any maximal ideal $\mathfrak{p}$ of A there is a unique maximal ideal $\mathfrak{m}$ of B such that $\mathfrak{m} \cap A = \mathfrak{p}$. [Hint: Use 2(b) above – there are also other ways.]

(b) Show that any proper nonzero ideal I of B is contained in finitely many maximal ideals of B. [Hint: Consider the maximal ideals of A that contain $I \cap A$ and use part (a)]

(a) Conclude that B is Noetherian (using the results of problems 2 and 3).

5. Let L and L' be finite extensions of $\mathbb{Q}$ and suppose that there exist $\alpha \in L$ and $\alpha' \in L'$ such that the integral closure of $\mathbb{Z}$ in L is $\mathbb{Z}[\alpha]$ and the integral closure of $\mathbb{Z}$ in L' is $\mathbb{Z}[\alpha']$. Suppose furthermore that $\Delta(\mathbb{Z}[\alpha]/\mathbb{Z}) + \Delta(\mathbb{Z}[\alpha']/\mathbb{Z}) = \mathbb{Z}$. Let M be the compositum LL' over K. Show that the integral closure of $\mathbb{Z}$ in M is $\mathbb{Z}[\alpha, \alpha']$. [Hint: 12.2 and 12.3 from October 13 are useful here.]

6. Exercise 2 from Janusz, page 57.
7. Exercise 3 from Janusz, page 58.
8. Exercise 4 from Janusz, page 58.
9. Exercise 5 from Janusz, page 58.
10. Exercise 7 from Janusz, page 58.