Math 430 Problem Set #7 Due 10/20/21

1. Let M be an $n \times n$ matrix over a field K. Suppose that $M^s = 0$ for some positive integer s. Show that the trace of M must be 0. [Hint: What are the eigenvalues of M?]
2. Let K be a field and let $B \supseteq K$ be an integral ring extension of K that is finitely generated as a K-module (B is not necessarily an integral domain). Then every element $x \in B$ gives rise to a multiplication map $r_x : b \mapsto xb$ from B to B. The map r_x is clearly a K-linear map and we can define its trace $\mathrm{T}_{B/K}(x)$ in the usual way. Show that the bilinear form $(x, y) = \mathrm{T}_{B/K}(xy)$ is degenerate whenever there exists an $x \in B$ such that $x \neq 0$ with $x^n = 0$ for some positive integer n.
3. Let A be a Dedekind domain with field of fractions K. Let L and L' be finite separable extensions of K and suppose that there exist $\alpha \in L$ and $\alpha' \in L'$ such that the integral closure of A in L is $A[\alpha]$ and the integral closure of A in L' is $A[\alpha']$. Let M be the compositum LL' over K. Is the integral closure of A in M necessarily equal to $A[\alpha, \alpha']$? Give a proof or a counterexample.
4. (Hard) Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt[3]{19})$.
5. Let α satisfy $\alpha^3 - \alpha^2 + 5$ and let $B = \mathbb{Z}[\alpha]$. Find all the primes $\mathcal{Q} \subset B$ lying above the following primes in $\mathbb{Z}$: 2, 3, 5, 7, 11.
6. Let a be a square-free positive integer. Let p be a prime number. Show that $\mathbb{Z}[\sqrt[p]{a}]$ is integrally closed if and only if $a^p - a \not\equiv 0 \pmod{p^2}$.
7. Let $L = K(\alpha)$ be separable and let F be the minimal monic for F over K. Show that for any $a \in K$, we have $N_{L/K}(a - \alpha) = F(a)$. [Hint: Factor F and use the fact that the norm of an element that generates L over K is the product of its conjugates.]
8. Let $(\cdot, \cdot)$ be the $\mathbb{Q}$-bilinear from on $\mathbb{Q}(\sqrt{5})$ given by $(a, b) = T_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(ab)$. Find the 2×2 matrix M such that

$$(a, b) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} M \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for $a = v_1 + v_2\sqrt{5}$ and $b = w_1 + w_2\sqrt{5}$. What is the determinant of M?