# Math 430 Problem Set #6 Due 10/13/21

1. Let $A$ be a DVR with maximal ideal $\mathcal{P}$, field of fractions $K$. Let $L$ be a finite separable extension of $K$ and let $B$ be a ring in $L$ that is integral over $A$ and has field of fractions $L$. Let $\mathcal{Q}$ be a prime in $B$ for which $\mathcal{Q} \cap A = \mathcal{P}$ and such that $\mathcal{Q}^e \supseteq \mathcal{P}$ and $[B/\mathcal{Q} : A/\mathcal{P}] = f$. Show that $\dim_{A/\mathcal{P}}(B/\mathcal{Q}^e) \geq ef$ with equality if and only if $B_{\mathcal{Q}}\mathcal{Q}$ is principal or $e = 1$.

2. Let $A$ be a DVR with maximal ideal $\mathcal{P}$, field of fractions $K$. Let $L$ be a finite separable extension of $K$ of degree $n$ and let $B$ be a ring in $L$ that is integral over $A$ and has field of fractions $L$. Suppose that $B\mathcal{P}$ factors as

$$B\mathcal{P} = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

Let $f_i$ be the relative degree $[B/\mathcal{Q}_i : A/\mathcal{P}]$ of $\mathcal{Q}_i$ over $\mathcal{P}$. Show that $\sum_{i=1}^{m} e_i f_i = n$ if and only if $B$ is Dedekind. [Hint: Recall that $\dim_{A/\mathfrak{p}} \mathfrak{p}/\mathfrak{p}^2 = 1$ if and only if $\mathfrak{p}$ is invertible.]

3. Let $p$ be a prime number. Show that $\mathbb{Z}[i]p$ factors as

$$\begin{array}{rcl} \mathcal{Q}^2 & ; & \text{if } p = 2 \\ \mathcal{Q}_1\mathcal{Q}_2 & ; & \text{if } p \equiv 1 \pmod 4 \\ \mathcal{Q} & ; & \text{if } p \equiv 3 \pmod 4, \end{array}$$

where $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2$ are primes of $\mathbb{Z}[i]$ and $\mathcal{Q}_1 \neq \mathcal{Q}_2$.

Also do the following problems from the book. You may assume that $L$ is separable over $K$ in all of them. (The notation with $e$ and $f$ is explained in the book.)

4. p. 29, Ex. 4
5. p. 30, Ex. 5
6. p. 32 Ex. 7