Math 430 Problem Set #5 Due 10/6/21

1. Let A be an integral domain such that (1) $A_{\mathfrak{m}}$ is Noetherian for every maximal ideal A and (2) every proper nonzero ideal of A is contained in at most finitely many maximal ideals. Show that A is Noetherian. [Hint: The beginning of Proposition 3.19 from the book provides a good model for this proof.]

2. Let $\theta = \sqrt{3}$ and let $R = \mathbb{Z}[\theta]$.
   (a) Write down a dual basis for the basis 1, θ for $\mathbb{Q}(\theta)$ over $\mathbb{Q}$ with respect to the bilinear form $(x, y) = \mathrm{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$.
   (b) Letting $R^\dagger$ denote the $\mathbb{Z}$-module generated by the dual basis above, find the order of the abelian group $R^\dagger/R$.

3. Let $\theta = \frac{1+\sqrt{5}}{2}$ and let $R = \mathbb{Z}[\theta]$.
   (a) Write down a dual basis for the basis 1, θ for $\mathbb{Q}(\theta)$ over $\mathbb{Q}$ with respect to the bilinear form $(x, y) = \mathrm{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$.
   (b) Letting $R^\dagger$ denote the $\mathbb{Z}$-module generated by the dual basis above, find the order of the abelian group $R^\dagger/R$.

4. Let K be a field. We define the resultant $\mathrm{Res}(f, g)$ as follows. Write

$$f(x) = b\prod_{i=1}^{m}(x - \alpha_i) \quad , \quad g(x) = c\prod_{j=1}^{n}(x - \beta_j),$$

with $b, c \in K^*$ and β_i, γ_j in some algebraic closure of K. Then

$$\mathrm{Res}(f, g) := b^n c^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i - \beta_j).$$

   (a) Let $h(x) = x^2 - 3$. Calculate $\mathrm{Res}(h(x), h'(x))$.
   (b) Let $t(x) = x^2 - x - 1$. Calculate $\mathrm{Res}(t(x), t'(x))$.
   (c) Compare your answers to 2(b) and 3(b).

5. Let f and g be as above and suppose that g is nonconstant. Show that

$$\mathrm{Res}(f(x), g(x)) = b^n \prod_{i=1}^{m} g(\alpha_i).$$

6. Let f be as above and assume that its leading coefficient b is 1 and that the degree of f is at least 2. Show that

$$\mathrm{Res}(f(x), f'(x)) = \prod_{\substack{1 \le i,j \le m \\ i \ne j}} (\alpha_i - \alpha_j).$$