# Problem Set #4 Due 9/29/21

1. Let $I$ be a nonzero ideal in a Noetherian integral $R$ domain of dimension 1. Show that $I$ factors as a product of primes if and only if $R_{\mathcal{M}}I$ is a power of $R_{\mathcal{M}}\mathcal{M}$ for every maximal ideal $\mathcal{M}$ of $R$. You may find Lemma 3.18 from the book to be helpful.

2. Let $d$ be a square-free integer congruent to 1 modulo 4 that is not congruent to 1 modulo 8. Let $R = \mathbb{Z}[\sqrt{d}]$. Let $I$ be the ideal generated by 2 in $R$, let $J$ be the ideal generated by $1 - \sqrt{d}$ in $R$, and let $\mathcal{P} = I + J$. Show that

(a) $R/\mathcal{P}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that $\mathcal{P}$ is therefore actually a prime;

(b) $\mathcal{P}^2 \subseteq I$;

(c) $R_{\mathcal{P}}/(R_{\mathcal{P}}I) \cong R_{\mathcal{P}}/(R_{\mathcal{P}}J)$ (as rings) but that $R_{\mathcal{P}}I \neq R_{\mathcal{P}}J$;

(d) $R_{\mathcal{P}}J$ is not a power of $R_{\mathcal{P}}\mathcal{P}$ and $J$ therefore cannot be factored as a product of prime ideals.

3. Let $R$ be an integral integral domain and let $M$ and $N$ be fractional ideals of $R$. Do the following:

(a) Show that if $M$ is invertible, then $(R : M)(R : N) = (R : MN)$.

(b) Let $R$ and $\mathcal{P}$ be as in problem #2. Show that $(R : \mathcal{P})(R : \mathcal{P}) \neq (R : \mathcal{P}^2)$.

4. Let $R$ be a Noetherian domain with the property that every *prime* ideal is principal. Show that every ideal of $R$ is principal. [Hint: You may want to begin by showing that $R$ is Dedekind]

5. We will say that a ring $R$ is unique factorization domain (UFD) if $R$ is an integral domain and if

- every nonunit $a \in R$ can be written as $\prod_{i=1}^{n} \pi_i^{e_i}$, where $e_i \in \mathbb{Z}^+$ and $R\pi_i$ is a prime ideal in $R$; and
- given two factorizations

$$a = \prod_{i=1}^{n} \pi_i^{e_i} = \prod_{i=1}^{m} \gamma_i^{f_i},$$

  where $e_i, f_i \in \mathbb{Z}^+$ and $R\pi_i$, $R\gamma_i$ are prime ideals in $R$, we must have $m = n$ and a reordering $\sigma$ of $1, \ldots, n$ such that $R\pi_i = R\gamma_{\sigma(i)}$ and $e_i = f_{\sigma(i)}$.

Since a principal ideal domain is a Dedekind domain or a field, it follows from unique factorization for ideals in a Dedekind domain that a principal ideal domain is a UFD. Show the partial converse: any Noetherian UFD of dimension 1 is a principal ideal domain. [You may use Problem # 4]

6. Let $R$ be a finite ring with $R \neq 0$. Show that $R$ is zero-dimensional.