Math 430 Problem set #2 Due September 15

1. (a) Suppose that α is integral of degree 2 over $\mathbb{Z}$ and that there is a basis m_1, m_2 for $\mathbb{Z}[\alpha]$ over $\mathbb{Z}$ such that

$$\begin{aligned} \alpha m_1 &= m_2 \\ \alpha m_2 &= -m_1 + m_2. \end{aligned}$$

Find the quadratic integral equation satisfied by α.

(b) Suppose that β is integral of degree 2 over $\mathbb{Z}$ and that there is a basis m_1, m_2 for $\mathbb{Z}[\beta]$ over $\mathbb{Z}$ such that

$$\begin{aligned} \beta m_1 &= m_2 \\ \beta m_2 &= 5m_1. \end{aligned}$$

Find the quadratic integral equation satisfied by β.

2. Let A be an unique factorization domain. Show that A is closed in its field of fractions. [Hint: Suppose an element a/b in the field of fractions of A in lowest terms (meaning that a and b have no common factor). Show that if b has a prime factor (i.e. is not a unit), then a/b cannot satisfy a monic polynomial f with coefficients in A by multiplying through by b^n and using the fact that a^n does not have this prime factor.]

3. Let A, B, and C be rings with $A \subseteq B \subseteq C$. Show that if B is integral over A and C is integral over B, then C is integral over A. (This was mentioned in class – it's worth going through the proof carefully.)

4. (Ex. 4, p. 7) Let d be a squarefree integer. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is

$$\mathbb{Z}[\sqrt{d}] \qquad \text{if} \quad d \equiv 2, 3 \pmod 4,$$

and

$$\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] \qquad \text{if} \quad d \equiv 1 \pmod 4.$$

5. (a) Let $\phi : A \longrightarrow B$ be a mapping of rings. Show that for any prime ideal $\mathcal{P}$ in B, the ideal $\phi^{-1}(\mathcal{P})$ is a prime ideal in A.

(b) Give an example of a surjective ring homomorphism $\phi : A \longrightarrow B$ for which there is a prime ideal $\mathcal{P}$ of A such that $\phi(\mathcal{P})$ is *not* a prime ideal.

6. (a) Give an example of a mapping of rings $\phi : A \longrightarrow B$ for which there is an ideal I of A such that $\phi(I)$ is not an ideal.

(b) Let $\phi : A \longrightarrow B$ be a surjective mapping of rings. Show that for any ideal I of A, the set $\phi(I)$ forms an ideal in B.

(c) Let $\phi : A \longrightarrow B$ be any mapping of rings. Show that for any ideal J of B, the set $\phi^{-1}(J)$ forms an ideal in A.

7. (Ex. 4, p.3) Show that if S is a multiplicative set not containing 0 in a Noetherian integral domain R, then $S^{-1}R$ is also a Noetherian integral domain.

8. (Continuation of 1 and 2 from last week.) Let R be a ring contained in $\mathbb{C}$. Suppose that for every $x \in R$, we have $|x|^2 \in \mathbb{Z}$, where $|\cdot|$ is the usual absolute value on $\mathbb{C}$. Suppose furthermore that for every $\alpha \in \mathbb{C}$, there is an $x \in R$ such that $|x - \alpha| < 1$. Show that R is a principal ideal domain. [Hint: Show that you have a Euclidean algorithm for the norm $N(r) = |r|^2$].

9. Use 8. to show that the rings $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ and $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ are both principal ideal domains.