Math 430 Problem Set #12 Due 12/1/21

1. Let d be any squarefree integer with $|d| > 1$ that is congruent to 3 (mod 4). Show that $L = \mathbb{Q}(\sqrt{d})$ admits a nontrivial extension that is unramified over all primes in $\mathcal{O}_L$. [Hint: Try adjoining i or $\sqrt{-d}$, both of which gave the same extension of L.]
2. Let L be a number field and let $y \in L$ have the property that $|\sigma(y)| = 1$ for all embeddings $\sigma : L \longrightarrow \mathbb{C}$. Show that y is a root of unity, i.e. that $y^n = 1$ for some $n > 1$. [Hint: Use the embedding $h : \mathcal{O}_L \longrightarrow \mathbb{R}^n$ used in the proof of the finitness of the class number and use the fact that $h(\mathcal{O}_L)$ is a full lattice, which thus contains at most finitely many elements in any bounded subset of $\mathbb{R}^n$.]
3. Compute the class group of $\mathbb{Z}[\frac{1+\sqrt{-31}}{2}]$.
4. Compute the class group of $\mathbb{Z}[\frac{1+\sqrt{-163}}{2}]$.

Recall that the p-adics $\mathbb{Q}_p$ are defined to be the completion of $\mathbb{Q}$ under the absolute value $|a|_p = e^{-v_p(a)}$ for $a \neq 0$ and v_p the discrete valuation corresponding to $\mathbb{Z}_{(p)}$ and $|0|_p = 0$. The set $\mathbb{Z}_p$ is the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ or alternately the set of $a \in \mathbb{Q}_p$ such that $|a|_p \leq 1$.

5. Show that there is no element $\alpha \in \mathbb{Q}_5$ such that $\alpha^2 = 5$. [Hint: What would $|\alpha|_5$ be? Use your answer to show there is no sequence of elements in $\mathbb{Q}_5$ that could converge to such an α.]

Recall that Hensel's lemma over the p-adics states that if $f \in \mathbb{Z}_p[x]$ is monic and $|f(\alpha)|_p < 1$ while $|f'(\alpha)|_p \geq 1$, then there is a $\gamma \in \mathbb{Q}_p$ such that $f(\gamma) = 0$ and $|\gamma - \alpha|_p < 1$.

6. Which of the following polynomials have roots in $\mathbb{Q}_3$? (Explain you answers):
   (a) $x^2 + 2$;
   (b) $x^3 + x + 1$;
   (c) $x^2 + 2x - 2$.