Math 430 Problem Set #1 Due September 1, 2021

1. For an element $u+vi$ of $\mathbb{Z}[i]$, define $N(u+vi) := u^2+v^2$.
(a) Show that for any $c, d \in \mathbb{Z}[i]$, we have $N(c)N(d) = N(cd)$.
(b) Show that for any elements $a, b \in \mathbb{Z}[i]$, it is possible to write

$$a = qb + r,$$

where $q, r \in \mathbb{Z}[i]$ and $N(r) < N(b)$.
(c) Conclude that $\mathbb{Z}[i]$ is a principal ideal domain. [Hint: Show that any ideal I is generated by a nonzero element of I having minimal norm.]

2. Use the Euclidean algorithm to show that $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is a principal ideal domain.

3. Answer each of the following yes or no and explain your answer.
(a) Is $11\sqrt{7}$ integral over $\mathbb{Z}$?
(b) Is $\frac{1+\sqrt{3}}{2}$ integral over $\mathbb{Z}$?
(c) Is $\frac{1+\sqrt{5}}{2}$ integral over $\mathbb{Z}$?
(d) Is $\mathbb{Z}[\sqrt{-19}]$ integrally closed in $\mathbb{Q}[\sqrt{-19}]$?

4. Show that ± 1 are the only units in the ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$.

5. It turns out that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a unique factorization domain (we will prove this later). Given this fact, find *all* integer pairs (x, y) such that $x^2 + 19 = y^3$ and justify your answers with a proof.