Math 430 Final Exam, Due December 15

1. Let $\mathcal{O}_L$ be ring of integers in a quadratic number field L. Let $p \mid \Delta(\mathcal{O}_L/\mathbb{Z})$. Show that

$$\left(\frac{\mathrm{N}_{L/\mathbb{Q}}(\alpha)}{p} \right) = 1$$

for every $\alpha \in \mathcal{O}_L$ such that p does not divide $|\mathrm{N}_{L/\mathbb{Q}}(\alpha)|$. [Hint: Write down the norm of an element $a + b\omega$ explicitly.]

2. Let $\mathcal{O}_L$ be ring of integers in a quadratic number field L. Suppose that there is some $p \mid \Delta(\mathcal{O}_L/\mathbb{Z})$ such that $p \equiv 3 \pmod 4$. Show that there does not exist any $\alpha \in \mathcal{O}_L$ such that $\mathrm{N}_{L/\mathbb{Q}}(\alpha) = -1$. Then give an example of a real quadratic field K (with no prime divisors that are congruent to 3 mod 4 of course) such that there is an $\alpha \in \mathcal{O}_K$ for which $N_{K/\mathbb{Q}}(\alpha) = -1$.

3. Show that $|x + y| = \max(|x|, |y|)$ for a non-Archimedean absolute value $|\cdot|$ on a field K when $|x| \neq |y|$. (Recall that a non-Archimedean absolute value $|\cdot|$ is an absolute value – multiplicative and satisfying the triangle inequality – with the property that $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$.

4. Suppose that $d < -7$ is a squarefree number that is congruent to $1 \pmod 8$. Show that $|\operatorname{Cl}[\mathbb{Z}[\frac{1+\sqrt{d}}{2}]| \neq 1$. [Hint: Look at one of the primes lying over 2].

5. Find the class group of $\mathbb{Z}[\sqrt{-22}]$. Show all of your work.

6. Find the class group of $\mathbb{Z}[\sqrt{-14}]$ (partial credit for getting its size, full credit for describing it up to isomorphism). Show all of your work.

7. Find the class group of $\mathbb{Z}[\frac{1+\sqrt{-23}}{2}]$. Show all of your work.

8. Find the class group of $\mathbb{Z}[\frac{1+\sqrt{-107}}{2}]$. Show all of your work.