An introduction to the MANDELBROT set, I

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MANDELBROT set

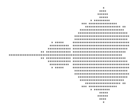


Figure: MANDELBROT set.

First published picture (1978);
Robert W. BROOKS and Peter MATELSKI.

Figure: Benoît MANDELBROT.

Definition of the MANDELBROT set

Quadratic polynomials: For $c \in \mathbb{C}$, $f_c(z) := z^2 + c$;

Viewed as a dynamical system acting on $\mathbb{C}$.

Iterates: For $n \geq 1$ integer, $f_c^n := \underbrace{f_c \circ \cdots \circ f_c}_{n}$;

n-th iterate of f_c.

MANDELBROT set: $\mathscr{M} := \{c \in \mathbb{C} : (f_c^n(0))_{n=1}^{\infty} \text{ is bounded}\}$.

Examples:

$c = 0$: $f_0(0) = 0 \Rightarrow (f_0^n(0))_{n=1}^{\infty}$ is constant $\Rightarrow 0 \in \mathscr{M}$;

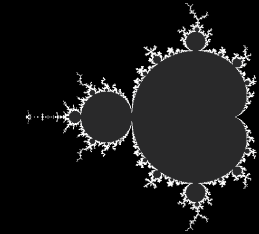
$c = 1$: $f_1(0) = 1$, $f_1^2(0) = 1^2 + 1 = 2$, $f_1^3(0) = 2^2 + 1 = 5$, ...

By induction $f_1^n(0) \geq n - 1 \Rightarrow 1 \notin \mathscr{M}$;

$c = i$: $f_i(0) = i$, $f_i^2(0) = i^2 + i = i - 1$, $f_i^3(0) = (i-1)^2 + i = -i$,
$f_i^4(0) = (-i)^2 + i = i - 1$.

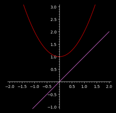
$f_i^4(0) = f_i^2(0) \Rightarrow (f_i^n(0))_{n=1}^{\infty}$ is eventually periodic $\Rightarrow i \in \mathscr{M}$.

MANDELBROT set



MANDELBROT set explorer.

Real trace of the MANDELBROT set



$c = 1$:

- The graph of f_1 is above the diagonal;
- $\Rightarrow (f_1^n(0))_{n=0}^{\infty}$ is increasing;
- $\Rightarrow f_1^n(0) \xrightarrow[n\to\infty]{} \infty$ or a fixed point of f_1;

Finite limit $x \Rightarrow f_1(x) = \lim_{n\to\infty} f_1^{n+1}(x) = x$.

- f_1 has no fixed point in $\mathbb{R}$

$\Rightarrow f_1^n(0) \xrightarrow[n\to\infty]{} \infty$.

For $c \in \mathbb{R}$: The graph of f_c is above the diagonal

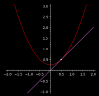
$\Leftrightarrow$ f_c has no fixed point in $\mathbb{R}$

$\Leftrightarrow$ $c > 1/4$.

Fixed point equation: $x^2 + c - x = 0$;
Discriminant: $1 - 4c$.

$(\frac{1}{4}, \infty) \cap \mathcal{M} = \emptyset$.

Real trace of the MANDELBROT set



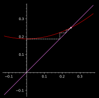
$c = \frac{1}{4}$:

- $x = \frac{1}{2}$ unique fixed point of $f_{\frac{1}{4}}$;
- $(f_{\frac{1}{4}}^n(0))_{n=0}^{\infty}$ increasing and $\to \frac{1}{2}$;

$x < \frac{1}{2} \Rightarrow f_{\frac{1}{4}}(x) < \frac{1}{2}$.

- $\Rightarrow \frac{1}{4} \in \mathcal{M}$.

Similar argument $\Rightarrow$ $[0, \frac{1}{4}] \subset \mathcal{M}$.



Graphical method to iterate.

Real trace of the MANDELBROT set

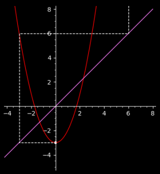


Figure: $c = -3$

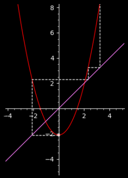


Figure: $c = -2.1$

Real trace of the MANDELBROT set

For $c \leq 1/4$: Fixed points of f_c,

$$\alpha(c) := \frac{1 - \sqrt{1-4c}}{2}, \beta(c) := \frac{1 + \sqrt{1-4c}}{2}.$$

- $x > \beta(c) \Rightarrow f_c(x) > x$;

The graph of f_c is above the diagonal on $(\beta(c), \infty)$.

- $\Rightarrow (f_c^n(x))_{n=1}^{\infty}$ is increasing and $\to \infty$.

There are no fixed points in $(\beta(c), \infty)$.

For $c = -3$ and -2.1: $f_c^2(0) > \beta(c) \Rightarrow c \notin \mathcal{M}$.
For $c < 0$: $(f_c^2(0) > \beta(c) \Leftrightarrow f_c(0) < -\beta(c) \Leftrightarrow c < -2) \Rightarrow c \notin \mathcal{M}$.

$(-\infty, -2) \cap \mathcal{M} = \emptyset$.

Real trace of the Mandelbrot set

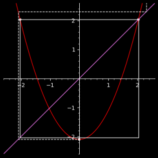


Figure: $c = -2.1$

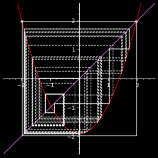


Figure: $c = -1.9$

For $c \in [-2, \frac{1}{4})$: $I(c) := [-\beta(c), \beta(c)]$ satisfies $f_c(I(c)) \subseteq I(c)$

$$\Rightarrow (f_c^n(0))_{n=1}^{\infty} \subset I(c) \Rightarrow c \in \mathscr{M}.$$

$$\boxed{\mathscr{M} \cap \mathbb{R} = [-2, \tfrac{1}{4}].}$$

Real trace of the Mandelbrot set

$$\boxed{\mathscr{M} \cap \mathbb{R} = [-2, \tfrac{1}{4}].}$$

Summary:

- $c > \frac{1}{4}$: $(f_c^n(0))_{n=1}^{\infty}$ is increasing and $\to \infty$;
- $c < -2$: $f_c^2(0) > \beta(c) \Rightarrow (f_c^n(0))_{n=2}^{\infty}$ is increasing and $\to \infty$;
- $c \in [-2, \frac{1}{4}]$: $f_c(I(c)) \subseteq I(c) \Rightarrow (f_c^n(0))_{n=1}^{\infty} \subset I(c) \Rightarrow c \in \mathscr{M}$.

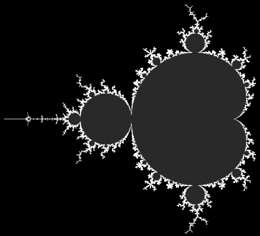
Remark

For $c \in [-2, \frac{1}{4})$: Up to a change of coordinates, f_c is the logistic map

$$g_\lambda(x) := \lambda x(1 - x), \text{ with } \lambda = 1 + \sqrt{1 - 4c}.$$

$h(x) := -\lambda(x - \frac{1}{2})$, $g_\lambda = h^{-1} \circ f_c \circ h$.

Hyperbolic components



Hyperbolic components

Periodic point of period n: $p \in \mathbb{C}$ such that $f_c^n(p) = p$.

- Orbit: $O(p) := \{p, f_c(p), \ldots, f_c^{n-1}(p)\}$;
- Multiplier: $|Df_c^n(p)|$;

Shared by every periodic point in $O(p)$.

- p is attracting if $|Df_c^n(p)| < 1$.

or hyperbolic attracting.

Theorem (Fatou)

For $c \in \mathbb{C}$ such that f_c has an attracting periodic point p: We have

$$\operatorname{dist}(f_c^n(0), O(p)) \xrightarrow[n\to\infty]{} 0.$$

Corollary

For $c \in \mathbb{C}$: f_c can have at most one attracting periodic orbit.

Hyperbolic components

Fatou's theorem $\Rightarrow$

$$\mathcal{H} := \{c \in \mathbb{C} : f_c \text{ has an attracting periodic point}\} \subset \mathcal{M}.$$

$\mathcal{H}$ is open;
For $c \in \mathcal{H}$, Smale's hyperbolicity theory applies to f_c.

Fatou Conjecture
$\mathcal{H}$ is dense in $\mathcal{M}$.

$\Rightarrow$ Hyperbolicity is dense in the quadratic family.

Hyperbolic component of $\mathcal{M}$: Connected component of $\mathcal{H}$.

Hyperbolic components

- α: fixed point of f_c;
- $\lambda := Df_c(\alpha)$.

$$c = \alpha - \alpha^2 = \frac{\lambda}{2} - \frac{\lambda^2}{4}.$$

$f_c(\alpha) = \alpha \Rightarrow c = \alpha - \alpha^2$;
$\lambda = Df_c(\alpha) = 2\alpha$.

For λ in $\mathbb{C}$:

$$\mu(\lambda) := \frac{\lambda}{2} - \frac{\lambda^2}{4}$$

unique $c \in \mathbb{C}$ such that f_c has a fixed point of multilier λ.

$$W_1 := \mu(\mathbb{D}) \subset \mathcal{M}.$$

Main hyperbolic component;
or hyperbolic component of period 1;
Main cardioid: ∂W_1.

Hyperbolic components

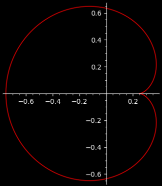


Figure: Main caridoid

Hyperbolic components

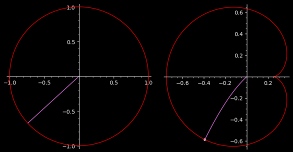


Figure: Internal ray of angle $\frac{\sqrt{5}-1}{2}$

Numerical experiments: Fatou's theorem.

Hyperbolic components

Equation of periodic points of period 2:

$$\frac{f_c^2(z) - z}{f_c(z) - z} = z^2 - z + c + 1.$$

p, $\widehat{p}$ periodic points of period 2,

$$\widehat{p} = f_c(p) \text{ and } Df_c^2(p) = Df_c(p)Df_c(f_c(p)) = 4p\widehat{p} = 4(c+1).$$

For λ in $\mathbb{C}$:

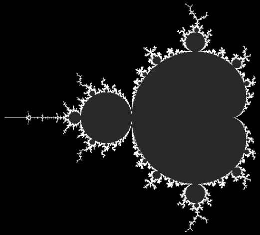
$$\nu(\lambda) := \frac{\lambda}{4} - 1$$

unique $c \in \mathbb{C}$ such that f_c has a periodic point of period 2 and multiplier λ.

$$W_2 := \nu(\mathbb{D}) \subset \mathcal{M}.$$

Hyperbolic component of period 2;
= Circle centered at -1 and radius $\frac{1}{4}$.

Hyperbolic components



Hyperbolic components

Summary:

- $\mathcal{H} := \{c \in \mathbb{C} : f_c \text{ has an attracting periodic point}\} \subset \mathcal{M}$;
  Hyperbolic component of $\mathcal{M}$: Connected component of $\mathcal{H}$.
- Main hyperbolic component or hyperbolic component of period one:

$$W_1 := \{c \in \mathbb{C} : f_c \text{ has an attracting fixed point}\} = \left\{\frac{\lambda}{2} - \frac{\lambda^2}{4} : \lambda \in \mathbb{D}\right\}.$$

- Hyperbolic component of period two:

$$W_2 := \{c \in \mathbb{C} : f_c \text{ has an attracting periodic point of minimal period } 2\} = \left\{\frac{\lambda - 1}{4} : \lambda \in \mathbb{D}\right\}.$$