CONFIGURATION SPACES AND BRAID GROUPS

FRED COHEN AND JONATHAN PAKIANATHAN

Abstract. The main thrust of these notes is 3-fold: (1) An analysis of certain $K(\pi, 1)$'s that arise from the connections between configuration spaces, braid groups, and mapping class groups, (2) a function space interpretation of these results, and (3) a homological analysis of the cohomology of some of these groups for genus zero, one, and two surfaces possibly with marked points, as well as the cohomology of certain associated function spaces.

An example of the type of results given here is an analysis of the space k particles moving on a punctured torus up to equivalence by the natural $SL(2, \mathbb{Z})$ action.

1. Introduction

The main thrust of these notes is 3-fold:
(1) An analysis of certain $K(\pi, 1)$-spaces that arise from the connections between configuration spaces, braid groups, and mapping class groups.
(2) A function space interpretation of these results, and
(3) A homological analysis of the cohomology of some of these groups for genus zero, one, and two surfaces (possibly with marked points).

These notes address certain properties of configuration spaces of surfaces, such as their connections to mapping class groups, as well as their connections to classical homotopy theory that emerged over 15 years ago. These constructions have various useful properties. For example, they give easy ways to compute the cohomology of certain related discrete groups, as well as give interpretations of this cohomology in terms of related mathematics.

In particular certain explicit models for Eilenberg-MacLane spaces of type $K(\pi, 1)$ are given for certain kinds of braid groups and mapping class groups. For example, we recall the classical result that configuration spaces of surfaces that are neither the 2-sphere nor the real projective plane are $K(\pi, 1)$ spaces. The analogous configuration spaces for the the 2-sphere and the real projective plane are not $K(\pi, 1)$'s, but this is remedied by considering natural actions of certain groups on these surfaces and forming the associated Borel constructions.

For example, the group $SO(3)$ acts on the 2-sphere by rotations. Hence, this group acts on the configuration space. Thus, there is an induced action of S^3, the non-trivial double cover of $SO(3)$, on these configuration spaces. A common theme throughout these notes is the structure of the Borel construction for these types of actions. For example, the case of the S^3 Borel construction for the natural action of S^3 on points in the 2-sphere or the projective plane give $K(\pi, 1)$ spaces whose fundamental group is the braid group of the 2-sphere or real projective plane respectively. The resulting spaces are elementary flavors of moduli spaces and may be described in elementary terms as spaces of polynomials. These were investigated in [].

One feature of the view here is that when calculations "work", they do so easily, and give global descriptions of certain cohomology groups. For example, the cohomology of the genus zero mapping class group with marked points gives a (sometimes) computable version of cyclic homology. Some concrete calculations are given.

Also, in genus one, there is a version of a based mapping class group. In this case the cohomology of these groups with all possible marked points admits an accessible and simple description. In additon, the genus 2 mapping class group has a very simple "configuration-like" description from which the torsion in the cohomology follows at once. Some of this has found application to the integral cohomology of $Sp(4,\mathbb{Z})$, the 4×4 integral matrices that preserve the symplectic form of $\mathbb{R}^4$. Most of the work here is directed at calculating torsion in the integral cohomology.

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2. Configuration spaces

Throughout these notes, we will assume X is a Hausdorff space and furthermore if X has a basepoint, we will assume it is non-degenerate. (This means that the inclusion of the basepoint into the space is a cofibration. We will discuss this more when we need it.) We will frequently also assume that X is of the homotopy type of a CW-complex.

We will use X^k to denote the Cartesian product of k copies of X. We will be interested in studying certain "configuration spaces" associated to X, so let us define these next.

Definition 2.1. *Given a topological space X, and a positive integer k, let*

$$F(X,k) = \{(x_1, \dots, x_k) \in X^k : x_i \neq x_j \text{ for } i \neq j\}.$$

This is the k-configuration space of X.

So we see that elements in the k-configuration space of X, correspond to k distinct, ordered points from X.

Now, it is easy to see that Σ_k, the symmetric group on k letters, acts on X^k by permuting the coordinates. If we restrict this action to $F(X,k)$, it is easy to check that it is free (no nonidentity element fixes a point). Thus we can form the quotient space

$$SF(X,k) = F(X,k)/\Sigma_k \tag{1}$$

and the quotient map $\pi : F(X,k) \to SF(X,k)$ is a covering map. An element of $SF(X,k)$ is a set of k distinct (unordered) points from X.

Remark 2.2. *We have of course that $F(X,1) = SF(X,1) = X$. (In these notes, $=$ will mean homeomorphic and we will use $\simeq$ to stand for homotopy equivalent).*

Remark 2.3. *It is easy to see that $F(-,k)$ defines a covariant functor from the category of topological spaces and continuous injective maps, to itself. This is not a homotopy functor. For example the unit interval, $[0,1]$, is homotopy equivalent to a point. However $F([0,1],2)$ is a nonempty space while $F(point,2)$ is an empty set.*

Definition 2.4. *Given a space X, we will find it convenient to let Q_m denote a set of m distinct points in X.*

Notice that for $k \geq m \geq 1$ there is a natural map

$$\pi_{k,m} : F(X,k) \to F(X,m)$$

obtained by projecting to the first m factors.

This map is very useful in studying the nature of $F(X,k)$ especially when X is a manifold without boundary. This is due to the following fundamental theorem:

Theorem 2.5 (Fadell and Neuwirth)**.** *If M is a manifold without boundary (not necessarily compact) and $k \geq m \geq 1$, then the map $\pi_{k,m}$ is a fibration with fiber $F(M - Q_m, k-m)$.*

We will use this theorem, to get our first insight into the nature of these configuration spaces.

Definition 2.6. *A $K(\pi,1)$-space X is a path connected space where $\pi_i(X) = 0$ for $i \geq 2$ and $\pi_1(X) = \pi$. It is well known, that the homotopy type of such a space is completely determined by its fundamental group (recall all our spaces are of the homotopy type of a CW complex).*

Let us first look at configuration spaces for 2-dimensional manifolds. Our first result, will be to show that most of these are $K(\pi,1)$-spaces.

Theorem 2.7. *Let M be either $\mathbb{R}^2$ or a closed 2-manifold of genus ≥ 1 (not necessarily orientable). $F(M-Q_m,k)$ has no higher homotopy for all $k \geq 1, m \geq 0$. In other words $F(M-Q_m,k)$ is a $K(\pi,1)$-space.*

Proof. We will prove it by induction on k. First the case $k=1$. If $m=0$, we just have to note that $M = F(M,1)$ is a $K(\pi,1)$-space and for $m \geq 1$, $M - Q_m \simeq$ bouquet of circles , and so is a $K(F,1)$-space where F is a free group of finite rank.

So we can assume $k > 1$ and that the theorem holds for all smaller values. By theorem 2.5, the map $\pi_{(k,1)} : F(M-Q_m,k) \to F(M-Q_m,1)$ is a fibration with fiber $F(M-Q_{m+1},k-1)$. However, by induction both the base and the fiber of this fibration are $K(\pi,1)$-spaces, and so by the homotopy long exact sequence of the fibration, we can conclude the total space is also a $K(\pi,1)$-space and so we are done.

□

It follows, from theorem 2.7, that for any closed 2-dimensional manifold M besides the sphere S^2 and the projective plane $\mathbb{R}P^2$, the homotopy type of $F(M,k)$ is completely determined by its fundamental group. So our next goal should be to understand that. It turns out there is quite a beautiful picture for the fundamental group of a configuration space in terms of braids and we shall explore this in the next section.

For now, let us state a lemma that can be used to ensure that a configuration space is path connected so that one does not have to worry about base points when talking about the fundamental group.

Lemma 2.8. *Let M be a connected manifold (without boundary) such that M remains connected when punctured at $k-1 \geq 0$ points. Then $F(M,k)$ is path connected. So in particular, if M is a connected manifold of dimension at least 2, all of its configuration spaces are path connected.*

Proof. The proof is by induction on k. When $k=1$, it follows easily from the hypothesis. So we can assume $k > 1$ and that we have proved it for smaller values. Then by theorem 2.5, $\pi_{k,1} : F(M,k) \to F(M,1)$ is a fibration with fiber $F(M-Q_1,k-1)$. Thus by induction, both the base and the fiber are path connected and hence so is the total space. □

3. Braid groups

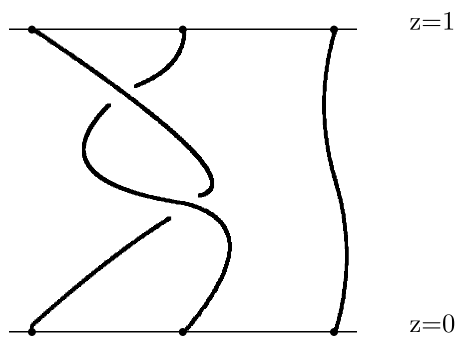
Let I denote the unit interval $[0,1] \subset \mathbb{R}$. If X is a space, we can view $X \times \{0\}$ as the bottom of the "cylinder" $X \times I$, and similarly $X \times \{1\}$ as the top.

Let I_k be the space which consists of k disjoint copies of I where the copies are labeled from 1 to k. Then let $0_i \in I_i$ be the point in the ith copy of I corresponding to 0 and similarly let $1_i \in I_i$ be the point in the ith copy of I corresponding to 1.

Let $E = (e_1, \ldots, e_k)$ be an element in $F(X,k)$, and let $\pi_I : X \times I \to I$ be the projection map to the second factor. We are now ready to define what we mean by a pure braid in X.

Definition 3.1. *A pure k-stranded braid in X (based at E) is a continuous, one to one map $f : I_k \to X \times I$ which satisfies:*
(a) $\pi_I \circ f : I_k \to I$ is the identity map on each component of I_k and
(b) $f(0_i) = (e_i, 0)$, $f(1_i) = (e_i, 1)$ for all $1 \leq i \leq k$.

Let us look at an example to make the formal definition above intuitively clear. Let us take $X = \mathbb{R}^2$ and $k = 3$. We can picture $X \times I$ as the subspace of $\mathbb{R}^3$ where the z-coordinate satisfies $0 \leq z \leq 1$. The following is a typical picture of a pure 3-stranded braid of $\mathbb{R}^2$.



It is obvious from the picture, that we would like to say two braids are equivalent if we can "deform" one to the other. Thus we define:

Definition 3.2. *Let f_0 and f_1 be two pure k-stranded braids based at E. Then we say f_0 is equivalent to f_1 if there exists a homotopy $F : I_k \times I \to X \times I$ between them, such that F restricted to $I_k \times \{t\}$ is a pure k-stranded braid for all $t \in I$.*

It is easy to see that the equivalence above, indeed gives an equivalence relation on the set of all pure k-stranded braids of X.

Now let us explore the natural correspondence between pure k-stranded braids based at E and loops in $F(X, k)$ based at E.

One can view such a loop as a map $\theta : I \to F(X, k)$ with $\theta(0) = \theta(1) = E$. However, $F(X, k) \subseteq X^k$ so we can take the components of the map θ to get maps $\theta_i : I \to X$ for $1 \leq i \leq k$. We can then define $f_i : I \to X \times I$ by

$$f_i(t) = (\theta_i(t), t).$$

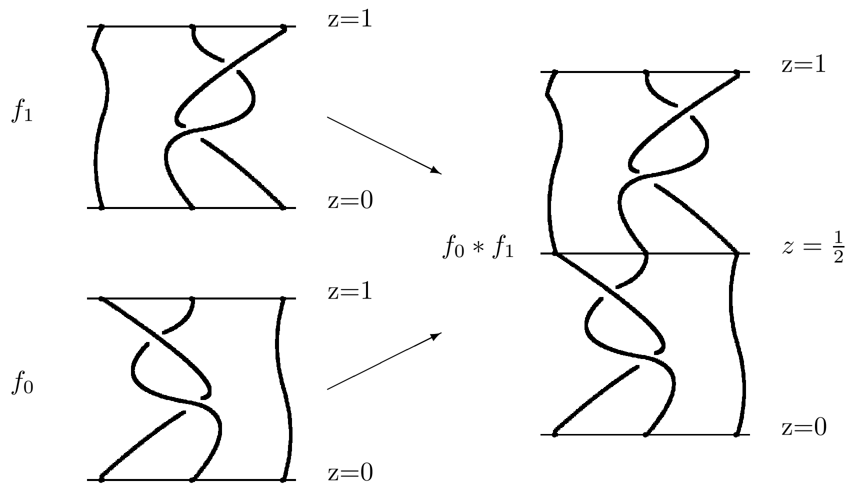
Finally we can take these maps and put them together to get a map $f : I_k \to X \times I$.

It is a routine exercise to check that the map f obtained is indeed a pure k-stranded braid based at E and that this establishes a one to one correspondence between loops in $F(X, k)$ based at E and pure k-stranded braids of X based at E.

Similarly, it is easy to check that two such loops are (base point preserving) homotopic if and only if the corresponding pure braids are equivalent. Thus we see a one to one corespondence between $\pi_1(F(X, k); E)$ and $PB_k(X; E)$, the set of equivalence classes of pure k-stranded braids in X, based at E.

Of course this implies that $PB_k(X; E)$ inherits the structure of a group from $\pi_1(F(X, k); E)$, but of course, we can also describe this multiplication naturally on the level of the braids.

Given two braids f_0 and f_1, we can think of f_0 as a braid between $X \times \{0\}$ and $X \times \{\frac{1}{2}\}$ and f_1 as a braid between $X \times \{\frac{1}{2}\}$ and $X \times \{1\}$, and then $f_0 * f_1$ is the braid obtained by stacking the braid f_1 on top of the braid f_0 as illustrated in the diagram below for the case $X = \mathbb{R}^2$ and $k = 3$. It also follows from the earlier correspondence, that the inverse of a pure braid is just obtained by turning the braid upside down.



Definition 3.3. *We will call $PB_k(X;E)$, the pure k-stranded braid group of X. We will usually surpress the basepoint E when it is obvious and write $PB_k(X)$.*

Remark 3.4. *Notice of course that $PB_1(X) = \pi_1(X)$ for any path connected space X.*

So we see that in general $\pi_1(F(X,k))$ can be interpreted as the pure k-stranded braid group $PB_k(X)$, and this will allow us to picture many relations among the elements of this group.

We saw, in section 2 that there are many examples where configuration spaces are $K(\pi,1)$-spaces. This means that in these cases, there will be a very strong connection between the configuration space and the corresponding pure braid group. We will begin to exploit this correspondence in the next section.

Definition 3.5. *$\pi_1(SF(X,k))$ is called the k-stranded braid group of X and is denoted $B_k(X)$.*

Recall that we have a covering map $\pi : F(X,k) \to SF(X,k)$, obtained by forming the quotient space under the free action of Σ_k on $F(X,k)$. Assuming that $F(X,k)$ is path connected, from covering space theory, this implies that we have the following short exact sequence of groups:

$$1 \to PB_k(X) \to B_k(X) \xrightarrow{\lambda} \Sigma_k \to 1$$

Fixing $E = (e_1, \dots, e_k) \in F(X,k)$, from covering space theory we can identify elements of $\pi_1(SF(X,k);\pi(E))$ with homotopy classes of paths in $F(X,k)$ starting at E and ending at a point of $F(X,k)$ in the same Σ_k-orbit of E. Using the same correspondence that associated loops in $F(X,k)$ with pure braids, we see that these paths are naturally associated to k-stranded braids of X which start at the points $\{e_1,\ldots,e_k\}$ at the bottom of $X \times I$ and end at the same set of points on the top, but where it is possible that the points get permuted.

It is easy to check that there is a well defined multiplication on the equivalence classes of these braids, just as before, which corresponds to the group structure of $\pi_1(SF(X,k))$. Thus $B_k(X)$ is just the group of k-stranded braids in X (based at some set of k distinct points) which are allowed to induce a permutation of the points. The map λ in the short exact sequence above is just the map that assigns to every braid, the permutation it induces on the k points. The pure braids are hence exactly the elements in the kernel of this map.

Definition 3.6. *$B_k(\mathbb{R}^2)$ is called Artin's k-stranded braid group and correspondingly, $PB_k(\mathbb{R}^2)$ is called Artin's k-stranded pure braid group. It follows from theorem 2.7 that*

$$F(\mathbb{R}^2,k) = K(PB_k(\mathbb{R}^2),1)$$

and

$$SF(\mathbb{R}^2,k) = K(B_k(\mathbb{R}^2),1).$$

4. Cohomology of groups

4.1. **Basic concepts.** This section is provided to recall the relevant basic facts about the cohomology of groups that we will need for the sequel. The reader is encouraged to read when these results are needed and used.

Recall for every "nice" topological group G (Lie groups and all groups equipped with the discrete topology are "nice"), we have a universal principal G-bundle,

$$EG \xrightarrow{\pi} BG$$

where EG is contractible and G acts freely on it. The isomorphism classes of principal G-bundles over any paracompact space X, are then in bijective correspondence to $[X, BG]$, the free homotopy classes of maps from X into BG. Furthermore, it is well known that BG is unique up to homotopy equivalence.

Furthermore the correspondence $G \to BG$ can be made functorial, so for every homomorphism of (topological) groups, $\lambda : G \to H$ we get a map $B(\lambda) : BG \to BH$. This map is always well defined up to free homotopy.

If G is discrete, then π is a covering map and BG is a $K(G,1)$-space. Since BG is unique up to homotopy equivalence, we may speak of the cohomology of a (discrete) group by defining

$$H^*(G;M) = H^*((K(G,1);M).$$

In general, we will also want to consider "twisted" coefficients in some G-module M.

One can also define the cohomology of a group, purely in the language of homological algebra.

Definition 4.1. *Given a group G, and a G-module M, we define*

$$H^*(G;M) = Ext^*_{\mathbb{Z}G}(\mathbb{Z},M)$$

and

$$H_*(G;M) = Tor_*^{\mathbb{Z}G}(\mathbb{Z},M)$$

where $\mathbb{Z}G$ is the integral group ring of G, and $\mathbb{Z}$ is given the structure of a G-module where every element of G acts trivially.

In other words, to calculate the cohomology of a group G, take any projective resolution of $\mathbb{Z}$ over the ring $\mathbb{Z}G$, apply $Hom_{\mathbb{Z}G}(-, M)$ to it, and calculate the cohomology of the resulting complex.

This definition, is equivalent to the topological one, see for example [B].

Remark 4.2. *It is an elementary fact that for any G-module M, one has*

$$H^0(G; M) = M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}.$$

M^G is called the group of invariants of the G-module M.

Remark 4.3. *It is also easy to see that one has*

$$H_0(G; \mathbb{Z}) = \mathbb{Z}$$

and

$$H_1(G; \mathbb{Z}) = G_{ab}$$

where G_{ab} is the abelianization of G.

4.2. **Examples.** Here are some basic calculations:

(a) Let $G = \mathbb{Z}/n$ be the cyclic group of order n. We can view $\mathbb{Z}/n$ as the nth roots of unity in $\mathbb{C}$ and have it act on $\mathbb{C}^\infty$ by coordinatewise multiplication. This action restricts to a free action on S^∞, the subspace of unit vectors whose coordinates are eventually zero and the quotient space under this action is an infinite Lens space which is a $K(\mathbb{Z}/n, 1)$.

One calculates:

$$H^k(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z}/n & \text{if } k > 0, k \text{ even} \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise .} \end{cases}$$

(b) let $G = F_n$ be a free group of rank n. In this case, a bouquet of n circles is a $K(F_n, 1)$. Since this is 1-dimensional, it is easy to see that $H^*(F_n; M) = 0$ for all $* > 1$ and any coefficient M. Furthermore, $H^1(F_n; \mathbb{Z})$ is a free abelian group of rank n.

4.3. **Cohomological dimension.** As we saw before, the free groups have the property that $H^*(G; M) = 0$ when $* > 1$, so we might say they have cohomological dimension 1. In general we define:

Definition 4.4. *The cohomological dimension of G is denoted $cd(G)$. It is the maximum dimension n such that $H^*(G; M) \neq 0$ for some G-module M. Of course, if there is no such maximum n, we say $cd(G) = \infty$.*

Thus we see the cohomological dimension of a nontrivial free group is one while $cd(\mathbb{Z}/n) = \infty$ for any $n \geq 2$.

Again, there is also a more direct description of cohomological dimension using homological algebra.

Definition 4.5. *Given a ring R, a projective resolution of a R-module M is a exact sequence of R-modules*

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

where the P_i are projective R-modules. If $P_n \neq 0$ and $P_k = 0$ for $k > n$ we say the resolution has length n.

Definition 4.6. *Given a ring R and an R-module M, we define $proj_R(M)$, the projective dimension of M, as the mimimum length of a projective resolution of M. Of course, it is possible $proj_R(M) = \infty$.*

Proposition 4.7. *For any group G,*

$$cd(G) = proj_{\mathbb{Z}G}(\mathbb{Z})$$

Proof. See [B]. □

Remark 4.8. *Given a G-module M, and a subgroup $H \leq G$, we can regard M as a H-module in the obvious way. It is easy to check that if M is a free (projective) $\mathbb{Z}G$-module, then M is also a free (projective) $\mathbb{Z}H$-module.*

Proposition 4.9. *For any group G, and subgroup $H \leq G$ we have $cd(H) \leq cd(G)$. Thus if $cd(G) < \infty$, G is torsion-free.*

Proof. It is easy to check that a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ restricts to a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. So the first statement follows. If G has nontrivial torsion it contains $\mathbb{Z}/n$ for some $n \geq 2$ and the second statement follows from the first as $cd(\mathbb{Z}/n) = \infty$. □

It is an elementary result that the only group that has cohomological dimension equal to zero is the trivial group. On the other hand we have seen that any nontrivial free group has cohomological dimension 1. It is a deep result of Stallings and Swan that the converse is also true, i.e.,

$$cd(G) \leq 1 \iff G \text{ is a free group.}$$

Thus we can think of $cd(G)$ as measuring how far a group G is from being free. If $cd(G) < \infty$, at least it is torsion-free, as we have seen.

We also have the following topological picture of cohomological dimension, given by the following proposition, whose proof can be found in [B].

Proposition 4.10. *Let the geometric dimension of G, denoted by $geod(G)$, be the minimum dimension of a $K(G,1)$-CW-complex. Then we have $geod(G) = cd(G)$ except possibly for the case where $cd(G) = 2$ and $geod(G) = 3$.*

Remark 4.11. *Whether the exceptional case $cd(G) = 2$ and $geod(G) = 3$ can occur is unknown. The conjecture that $cd(G) = geod(G)$ in general is known as the Eilenberg-Ganea conjecture.*

We will also need the following basic theorem on subgroups of finite index whose proof can be found in [B].

Theorem 4.12. *Let G be a torsion-free group and H a subgroup of finite index. Then $cd(G) = cd(H)$.*

4.4. FP_∞ groups.

Definition 4.13. *A group G is of type FP_n if there is a projective resolution $\{P_i\}_{i=0}^{\infty}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ such that P_j is finitely generated for $j = 0, \ldots, n$. A group G is of type FP_∞ if it is of type FP_n for all n.*

It is elementary to show that every group is FP_0 and that a group is of type FP_1 if and only if it is finitely generated. So we can think of FP_n for higher n as strengthenings of the condition that G be finitely generated.

It is also true that if G is finitely presented, then G is of type FP_2. For a long time, it was conjectured the converse was true but Bestvina and Brady provided a counterexample - a group that is FP_2 but not finitely presented.

Remark 4.14. *It follows immediately, that if G is of type FP_n then for any G-module M, which is finitely generated as an abelian group, we have $H_i(G; M)$ and $H^i(G; M)$ finitely generated for all $i \leq n$.*

One has the following topological picture:

Proposition 4.15. *A finitely presented group G is of type FP_n if and only if there exists a $K(G, 1)$-CW-complex such that the n-skeleton is finite. If G is finitely presented, then it is of type FP_∞ if and only if there exists a $K(G, 1)$-CW-complex such that the n-skeleton is finite for all n.*

Definition 4.16. *A group is of type FP if it is of type FP_∞ and has finite cohomological dimension. This happens exactly when there is a projective resolution $\{P_i\}_{i=0}^{\infty}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ which has finite length and such that each P_i is finitely generated.*

If we are given such a finite projective resolution, it does not necessarily mean that we can find a finite length resolution using free modules of finite rank. Thus we define:

Definition 4.17. *A group is of type FL if there is a resolution of finite length for $\mathbb{Z}$ over $\mathbb{Z}G$ using finitely generated free modules.*

Again, we have a more concrete topological picture.

Proposition 4.18. *Let G be a finitely presented group. Then G is of type FP if and only if there exists a finitely dominated $K(G, 1)$-CW-complex. Similarly, G is of type FL if and only if there exists a finite $K(G, 1)$-CW complex.*

Thus for example F_n is FL as we can take $K(F_n, 1)$ to be the bouquet of n circles.

We will also find the following proposition useful:

Proposition 4.19. *If we have a short exact sequence of groups*

$$1 \to \Gamma_0 \to \Gamma \to \Gamma_1 \to 1$$

then if Γ_0 and Γ_1 are of type FP (respectively FL) then so is Γ.

Definition 4.20. *If a group G is of type FP, we define $\chi(G)$, the Euler characteristic of G to be the Euler characteristic of a $K(G, 1)$. This makes sense since $H^*(G; \mathbb{Z})$ is finitely generated in each dimension, and is zero for $* > cd(G)$.*

The following proposition recalls well-known facts about subgroups of finite index.

Proposition 4.21. *If G is a torsion-free group, and H is a subgroup of finite index. Then G is of type FP if and only if H is. Furthermore in this case,*

$$\chi(H) = |G : H|\chi(G).$$

4.5. **The five lemma.** We state a refined form of the five lemma here as a convenient quick reference. The reader can supply the usual diagram chasing proof if they feel inclined!

Lemma 4.22. *Given a commutative diagram of (not neccessarily abelian) groups and homomorphisms:*

$$\begin{array}{ccccccccc} H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & H_4 & \longrightarrow & H_5 \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \downarrow{\scriptstyle \mu_4} & & \downarrow{\scriptstyle \mu_5} \\ G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & G_4 & \longrightarrow & G_5 \end{array}$$

where the rows are exact, the following is true:

(a) If μ_2, μ_4 are monomorphisms and μ_1 is an epimorphism then μ_3 is a monomorphism.

(b) If μ_2, μ_4 are epimorphisms and μ_5 is a monomorphism then μ_3 is an epimorphism.

(c) If μ_1, μ_2, μ_4 and μ_5 are isomorphisms, then so is μ_3.

4.6. **The Lyndon-Hochschild-Serre (LHS) spectral sequence.** For any short exact sequence of groups

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$$

and G-module M, we have a E_2-spectral sequence abutting to $H^*(G; M)$ and whose E_2 term is given by

$$E_2^{p,q} = H^p(Q; H^q(N; M)).$$

Of course, there is also a similar spectral sequence in homology. (Here, recall that in general, $H^q(N; M)$ is a nontrivial Q-module where the Q action is induced by the conjugation action of G on N.)

This spectral sequence has a topological origin, for if you have such an exact sequence of groups, then

$$BN \xrightarrow{Bi} BG \xrightarrow{B\pi} BQ$$

is a fibration, and the LHS-spectral sequence is nothing more than the Serre spectral sequence for this fibration, possibly using twisted coefficients.

5. Polyfree groups

Definition 5.1. *A normal series for a group G is a sequence of subgroups*

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

where each G_i is normal in G. $\Gamma_i = G_i/G_{i-1}$ is referred to as the ith factor for $1 \leq i \leq n$. The length of a normal series is the number of nontrivial factors.

Definition 5.2. *A polyfree group is a group which has a normal series where all the factors are finitely generated free groups. Such a normal series will be refered to as a polyfree series and the rank of the kth factor will be called d_k, the kth exponent.*

Remark 5.3. *Apriori, we do not know whether every polyfree series of a given polyfree group will have the same length, nor do we know if the exponents may vary with the polyfree series chosen. We will return to this point shortly.*

Our interest in polyfree groups stems from the fact that many of the pure braid groups we have considered up to now, are polyfree.

Theorem 5.4. *Let M be equal to $\mathbb{R}^2$ or a closed 2-manifold of genus $g \geq 1$. Then $PB_k(M - Q_m)$ is polyfree for any $m, k \geq 1$. (Notice that we do have to puncture the manifold at least once in general). Furthermore there is a polyfree series of length k where the rank of the ith factor is equal to $2g + m - 1 - i + k$ for $1 \leq i \leq k$. (These formulas also work for $M = \mathbb{R}^2$ if we set $g = \frac{1}{2}$).*

Proof. As usual, we will induct on k. For $k = 1$, $F(M - Q_m, k) = M - Q_m$ is a $K(F_{2g+m-1}, 1)$ where F_{2g+m-1} is a free group of rank $2g+m-1$. Thus the theorem follows easily in this case. So we may assume $k > 1$ and that the theorem is proven for smaller values of k. Then by theorem 2.5, we have

$$\pi_{k,k-1} : F(M - Q_m, k) \to F(M - Q_m, k - 1)$$

is a fibration with fiber $F(M - Q_{m+k-1}, 1)$. Since these spaces have trivial π_2 by theorem 2.7, we get a short exact sequence of groups

$$1 \to F_{2g+m-2+k} \to PB_k(M - Q_m) \to PB_{k-1}(M - Q_m) \to 1.$$

By induction $PB_{k-1}(M - Q_m)$ has a polyfree series of length $k-1$ with the claimed exponents and so it follows from the short exact sequence of groups above, that $PB_k(M - Q_m)$ has a polyfree series of length k. It is an easy exercise which will be left to the reader, to check that the factors have the ranks claimed. □

Remark 5.5. *For $k \geq 2$, by considering the map $\pi_{k,1} : F(\mathbb{R}^2, k) \to F(\mathbb{R}^2, 1) = \mathbb{R}^2$ we see that*

$$F(\mathbb{R}^2, k) = \mathbb{R}^2 \times F(\mathbb{R}^2 - Q_1, k - 1)$$

so it follows that $PB_k(\mathbb{R}^2) = PB_{k-1}(\mathbb{R}^2 - Q_1)$ and so the pure k-stranded Artin braid group is polyfree with a polyfree series of length $k - 1$ given by the theorem above.

So now we have a lot of motivation to study the class of polyfree groups. Let us begin with some elementary results. First recall the following classical theorem of P. Hall:

Theorem 5.6 (P. Hall)**.** *If*

$$1 \to N \to G \to Q \to 1$$

is a short exact sequence of groups, with N and Q finitely presented, then G is finitely presented.

Proof. See e.g. [R] □

Proposition 5.7. *Suppose G is a polyfree group with a polyfree series of length n. Then:*
(a) $cd(G) \leq n$ and in particular G is torsion-free.
(b) G is of type FL.
(c) G is finitely presented.
(d) If the exponents for the polyfree series satisfy $d_k \geq 2$ for all $1 \leq k \leq n$ then G has trivial center.

Proof. We will induct on n, the length of the polyfree series. If $n = 1$, then G is free of rank d_1. The theorem then follows readily once we note that a free group of rank $d_1 \geq 2$ has trivial center. (Since for example it is a nontrivial free product.)

So we can assume $n > 1$ and that we have proved the proposition for all smaller n. Let $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ be the given polyfree series. Notice we have a short exact sequence

$$1 \to G_{n-1} \to G \xrightarrow{\pi} F_{d_n} \to 1$$

and that G_{n-1} is polyfree with a series of length $n-1$. Thus by induction G_{n-1} is FL and finitely presented and hence so is G by theorem 5.6 and proposition 4.19. So we have proven (b) and (c).

For (d), if we assume $d_k \geq 2$ for all k then by induction G_{n-1} has trivial center and of course so does F_{d_n}. Now if $c \in G$ is a central element, then it is easy to see that $\pi(c)$ will be central in F_{d_n} and hence trivial. Thus $c \in G_{n-1}$ and so c is a central element in G_{n-1} which finally lets us conclude $c = 1$. Thus we have proven (d).

So it remains only to prove (a). By induction, $cd(G_{n-1}) \leq n-1$. Now suppose M is any G-module. Then we can apply the LHS-spectral sequence to the short exact sequence above. Now $E_2^{p,q} = H^p(F_{d_n}; H^q(G_{n-1}; M))$ is zero if either $p > 1$ or $q > n-1$, as $cd(G_{n-1}) \leq n-1$. Thus we see there can be no nontrivial differentials in this spectral sequence and so $E_2 = E_\infty$.

However this spectral sequence converges to $H^*(G; M)$ and so we can conclude easily that $H^*(G; M) = 0$ for $* > n$. Since this holds for any G-module M, we can conclude $cd(G) \leq n$. ☐

Remark 5.8. *Given a polyfree group G with a polyfree series of length n, proposition 5.7 guarantees the existance of a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of length less than or equal to n, using finitely generated, free $\mathbb{Z}G$-modules.*

A nice resolution of this sort was constructed in [DCS] *using Fox free derivatives in the case where the polyfree group satisfies an extra splitting condition which we will consider later. The naturality of this resolution was also exploited to recover many interesting representations which we will also look at later.*

Next we consider subgroups of finite index in a polyfree group.

Proposition 5.9. *Let G be a polyfree group with a polyfree series of length n and let H be a subgroup of finite index in G. Then H is polyfree with a polyfree series of length n.*

Proof. We proceed by induction on n. If $n = 1$ then G is a nontrivial free group of finite rank. It follows then from the Nielsen-Schreier theorem (see [R]) that H is also free of finite rank and so this case follows.

So we may assume $n > 1$ and that we have proved the proposition for smaller values. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be the polyfree series for G. Then if we set $H_i = G_i \cap H$ for $0 \leq i \leq n$, it is easy to check $1 = H_0 < H_1 < \cdots < H_n = H$ is a normal series for H. Furthermore, if $H_i/H_{i-1} \to G_i/G_{i-1}$ is the map induced by inclusion of H in G, it is easy to check this map is a well defined monomorphism (injection) of groups. On the other hand the image of this map has finite index in G_i/G_{i-1} as G_i/H_i injects (as sets) into the finite set G/H via the map induced by inclusion. Thus we see each H_i/H_{i-1} can be viewed as a subgroup of finite index in the nontrivial finitely generated free group G_i/G_{i-1} and hence is itself a nontrivial free group of finite rank, again by the Nielsen-Schreier theorem. Thus $1 = H_0 < H_1 < \cdots < H_n = H$ is a polyfree series for H of length n. □

Remark 5.10. *An arbitrary subgroup of a free group of finite rank need not have finite rank and so we see that arbitrary subgroups of a polyfree group need not be polyfree.*

Now let us show that for any given polyfree group G, all polyfree series for G have the same length. First we will need a technical lemma. Let $\mathbb{F}_2$ be the field with two elements.

Lemma 5.11. *Let G be a polyfree group with a polyfree series $1 = G_0 < G_1 < \cdots < G_n = G$ of length n. Then there exists a subgroup T of finite index such that $H^n(T;\mathbb{F}_2) \neq 0$. Furthermore, T contains G_1 and $\chi(T) = \chi(G_1)\chi(T/G_1)$. Thus $\chi(G) = \chi(G_1)\chi(G/G_1)$.*

Proof. As usual we will proceed by induction on n. If $n = 1$, G is a nontrivial free group of finite rank. Thus we can take $T = G = G_1$ as $H^1(G;\mathbb{F}_2) = Hom(G;\mathbb{F}_2)$ is nonzero. (The statement about Euler characteristics follows as $\chi(1) = 1$.)

So we can assume $n > 1$ and that we have proved the lemma for smaller n. Then notice that G/G_1 has a polyfree series of length $n-1$.

Now G acts by conjugation on G_1 and hence on the finite vector space

$$H^1(G_1;\mathbb{F}_2) = Hom(G_1,\mathbb{F}_2).$$

If we let K be the kernel of this action on $H^1(G_1;\mathbb{F}_2)$, then K is normal and of finite index in G and furthermore $G_1 \subseteq K$.

Now $K/G_1 \subseteq G/G_1$ is a subgroup of finite index and so by proposition 5.9, K/G_1 is polyfree with a polyfree series of length $n-1$. So, by hypothesis, we may find T/G_1 with $H^{n-1}(T/G_1;\mathbb{F}_2) \neq 0$ and T a subgroup of K of finite index. However $T/G_1 \subseteq G/G_1$ and so $cd(T/G_1) \leq n-1$ by proposition 5.7. So if we look at the LHS-spectral sequence for the short exact sequence

$$1 \to G_1 \to T \to T/G_1 \to 1$$

we see that $E_2^{p,q} = H^p(T/G_1; H^q(G_1;\mathbb{F}_2))$ is zero if $p > n-1$ or $q > 1$ and $E_2^{n-1,1}$ is nonzero. (Here we use that $T \subseteq K$ so that $E_2^{*,*} = H^*(T/G_1;\mathbb{F}_2) \otimes H^*(G_1;\mathbb{F}_2)$.) It is easy to see that we must have $E_2^{n-1,1} = E_\infty^{n-1,1}$ and so we can conclude $H^n(T;\mathbb{F}_2)$ is nonzero. T is obviously of finite index in G so it only remains to prove the statement about $\chi(T)$.

In the spectral sequence above, the E_2 term has Euler characteristic

$$\chi(G_1)\chi(T/G_1).$$

On the other hand the E_∞ term abuts to $H^*(T;\mathbb{F}_2)$ which has Euler characteristic $\chi(T)$ and so the equality $\chi(T) = \chi(G_1)\chi(T/G_1)$ follows from the general fact that the Euler characteristic remains constant on each page of the LHS-spectral sequence. Thus our induction on n goes through.

Finally the equality $\chi(G) = \chi(G_1)\chi(G/G_1)$ follows from the previous one which had T instead of G, theorem 4.21, and the fact that the index of T in G is the same as the index of T/G_1 in G/G_1.

□

Proposition 5.12. *Let G be a polyfree group with a polyfree series of length n and exponents d_k, $1 \leq k \leq n$. Then $cd(G) = n$ and so every polyfree series of G has length n. Furthermore $\chi(G) = \prod_{k=1}^{n}(1 - d_k)$.*

Proof. By proposition 5.7, we have $cd(G) \leq n$. On the other hand, by lemma 5.11, we have a subgroup T of G of finite index, such that $cd(T) \geq n$. Thus the first part follows as $cd(G) = cd(T)$ by proposition 4.12.

Now let us prove the formula for $\chi(G)$ by induction on n. For $n = 1$, it follows as a bouquet of d_1 circles is a $K(G, 1)$. So as usual we can assume $n > 1$ and that we have the formula for smaller n.

Again by lemma 5.11, we have $\chi(G) = \chi(G_1)\chi(G/G_1) = (1 - d_1)\chi(G/G_1)$. The formula now follows readily once we observe that G/G_1 has a polyfree series of length $n - 1$ with exponents $d_2, \ldots, d_n$ and so by induction,

$$\chi(G/G_1) = \prod_{k=2}^{n}(1 - d_k).$$

□

Remark 5.13. *So from what we have seen, all polyfree series for a given polyfree group G, have the same length which is equal to $cd(G)$. However the exponents of different polyfree series need not be the same.*

For example we saw that $PB_4(\mathbb{R}^2) = PB_3(\mathbb{R}^2 - Q_1)$ is polyfree with exponents $(3, 2, 1)$ by theorem 5.4. However it is known that this group also admits a polyfree series with exponents $(5, 2, 1)$ (see [DCS]).

However $\prod_{k=1}^{n}(1 - d_k)$ is independent of the polyfree series chosen as it is equal to $\chi(G)$ as we have seen.

6. Configuration spaces for the 2-sphere

We have seen in theorem 2.7 that for any closed 2-dimensional manifold M besides the sphere S^2 and the projective plane $\mathbb{R}P^2$, $F(M, k)$ is a $K(PB_k(M), 1)$. Thus the configuration space is a good model for the pure braid group of M. Now let us see what we can say in the case where M is S^2 or $\mathbb{R}P^2$. Let us first look at S^2. First note that $SO(3)$ acts naturally on $\mathbb{R}^3$ and this action preserves the unit sphere S^2. Hence $SO(3)$ acts naturally on $F(S^2, k)$ for any k via a diagonal action. We will describe these configuration spaces now up to homotopy equivalence (which will be denoted by $\simeq$). In fact we will find homotopy equivalences which preserve the $SO(3)$ actions.

Proposition 6.1.

$$F(S^2, k) \simeq \begin{cases} S^2 & \text{if } k = 1, 2 \\ SO(3) & \text{if } k = 3 \\ SO(3) \times F(S^2 - Q_3, k - 3) & \text{if } k > 3. \end{cases}$$

Furthermore, there are homotopy equivalences which are equivariant with respect to the natural $SO(3)$ actions on $F(S^2, k)$ and the action of $SO(3)$ on itself by left multiplication. (Here $SO(3)$ acts on $SO(3) \times F(S^2 - Q_3, k - 3)$ by acting entirely on the left factor via left multiplication.)

Proof. The $k = 1$ case is trivial so let us look at the $k = 2$ case first. By theorem 2.5, $\pi : F(S^2, 2) \to F(S^2, 1) = S^2$ is a fibration with contractible fiber $F(S^2 - Q_1, 1) =$

R^2. Thus π is a homotopy equivalence and it is trivial to see it is equivariant with respect to the natural actions of $SO(3)$ on $F(S^2,2)$ and S^2.

Now let us look at the case where $k=3$. Again by theorem 2.5, there is a fibration

$$\pi : F(S^2,3) \to S^2$$

obtained by projecting onto the first factor.

Fix a point $p_1 \in S^2$. Let $F = \pi^{-1}(p_1)$, the fiber over the point p_1. Then again by theorem 2.5, since $F = F(S^2 - Q_1, 2)$ there is a fibration

$$\hat{\pi} : F \to (S^2 - Q_1) = \mathbb{R}^2$$

with fiber $S^2 - Q_2$, obtained by projecting onto a factor.

Since the base space of the fibration $\hat{\pi}$ is contractible, we conclude that

$$F = \mathbb{R}^2 \times (S^2 - Q_2) \simeq S^1.$$

Now we will define a map $\lambda : SO(3) \to F(S^2,3)$. Fix $\bar{p} = (p_1,p_2,p_3) \in F(S^2,3)$. Specifically, we will take $p_1 = (0,0,1), p_2 = -p_1$ and $p_3 = (1,0,0)$. Then we can set

$$\lambda(\alpha) = (\alpha(p_1), \alpha(p_2), \alpha(p_3))$$

for all $\alpha \in SO(3)$.

It is easy to see that λ is a $SO(3)$-equivariant, continuous map from $SO(3)$ to $F(S^2,3)$. We now want to show it is a homotopy equivalence. Let I denote the isotropy group of the point $p_1 \in S^2$ under the $SO(3)$ action. Then I is isomorphic to $SO(2)$ as a topological group and $\lambda|_I$ maps I into F.

Claim 6.2. *The map $\lambda|_I : I \to F$ is a homotopy equivalence.*

Proof. The action of I on S^2 is given by rotation about the z-axis fixing p_1 which can be thought of as the north pole of S^2. Notice that since we chose $p_2 = -p_1$, any element of I will also fix p_2 (which is the south pole). Recall we had a fibration $\hat{\pi} : F \to S^2 - \{p_1\}$ and let us set $F' = \hat{\pi}^{-1}(p_2)$, the fiber above the point p_2. We saw above that $F' = S^2 - \{p_1,p_2\} \simeq S^1$ and that the inclusion of F' into F is a homotopy equivalence.

Now notice, that $\lambda|_I$ actually maps I into F' since any element of I fixes p_2. If we let α_θ denote the element of I which corresponds to rotation through an angle θ, then the loop $\{\alpha_\theta : 0 \le \theta \le 2\pi\}$ gets mapped by λ to the generator of $\pi_1(F')$. Thus we see that $\lambda|_I : I \to F'$ is a homotopy equivalence since both I and F' are homotopic to S^1 and λ is surjective on the π_1 level. Hence $\lambda|_I : I \to F$ is a homotopy equivalence since the inclusion of F' into F is a homotopy equivalence. $\square$

Now one can check easily that we get the following commutative diagram:

$$\begin{array}{ccccc}
I & \longrightarrow & SO(3) & \xrightarrow{f} & S^2 \\
\downarrow{\lambda|_I} & & \downarrow{\lambda} & & \downarrow{Id} \\
F & \longrightarrow & F(S^2,3) & \xrightarrow{\pi} & S^2
\end{array}$$

where Id is the identity map and $f(\alpha) = \alpha(p_1)$ for $\alpha \in SO(3)$. Furthermore, each row is a fibration. Comparing the long exact sequence in homotopy for the two fibrations, one sees that the vertical maps induce an isomorphism on homotopy groups on the base and fiber levels of these fibrations and hence (by the five lemma) also on the level of total spaces. Thus, $\lambda : SO(3) \to F(S^2,3)$ induces an isomorphism on homotopy groups and hence is a homotopy equivalence and so we have the result for the case $k = 3$.

Now for the final case when $k > 3$. Let $\bar{p} = (p_1, \dots, p_k) \in F(S^2, k)$ extend (p_1, p_2, p_3) chosen in the previous case. Then again we can define a map

$$\lambda : SO(3) \times F(S^2 - \{p_1, p_2, p_3\}, k - 3) \to F(S^2, k)$$

by

$$\lambda(\alpha, (a_4, \dots, a_k)) = (\alpha(p_1), \alpha(p_2), \alpha(p_3), \alpha(a_4), \dots, \alpha(a_k))$$

for $\alpha \in SO(3)$ and $(a_4, \dots, a_k) \in F(S^2 - \{p_1, p_2, p_3\}, k - 3)$.

It is easy to see that this map is continuous and $SO(3)$-equivariant under the $SO(3)$-actions described in the statement of this lemma. Furthermore, we get the following commutative diagram:

$$\begin{array}{ccccc}
\pi^{-1}(Id) & \longrightarrow & SO(3) \times F(S^2 - \{p_1, p_2, p_3\}, k - 3) & \xrightarrow{\pi} & SO(3) \\
{\scriptstyle Id}\big\downarrow & & {\scriptstyle \lambda}\big\downarrow & & {\scriptstyle \lambda}\big\downarrow \\
\pi_{k,3}^{-1}(p_1, p_2, p_3) & \longrightarrow & F(S^2, k) & \xrightarrow{\pi_{k,3}} & F(S^2, 3)
\end{array}$$

where as usual Id stands for an identity map and π is projection onto the first factor. Furthermore both rows are fibrations. We have seen that the vertical map on the base level is a homotopy equivalence and also trivially the vertical map on the fiber level is also a homotopy equivalence and so it follows that λ is indeed a homotopy equivalence (by looking at the long exact sequences in homotopy for the fibrations, and using the five-lemma). So this concludes the proof of the lemma. $\square$

Proposition 6.1 has the following immediate corollary:

Corollary 6.3.

$$PB_k(S^2) = \begin{cases} 1 & \textit{if } k = 1, 2 \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } k = 3 \\ \mathbb{Z}/2\mathbb{Z} \times PB_{k-3}(\mathbb{R}^2 - Q_2) & \textit{if } k > 3 \end{cases}$$

Notice that $F(S^2, k)$ is never a $K(\pi, 1)$-space due to the higher homotopy in $SO(3)$ and S^2. However, we will see in the next subsection, that one can perform a natural construction to $F(S^2, k)$ for $k \geq 3$ and obtain a useful $K(\pi, 1)$-space.

6.1. **The Borel Construction.** We will now recall a very important construction. First some elementary definitions:

Definition 6.4. *If G is a topological group with identity element e, then a (left) G-space is a space X together with a continuous map $\mu : G \times X \to X$ which satisfies:*
(a) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and
(b) $e \cdot x = x$
for all $g_1, g_2 \in G$, $x \in X$. (Here we use the notation $g \cdot x$ for $\mu(g, x)$ and e denotes the identity element of G). Of course, similarly there is a notion of right G-space where the action is on the right instead of on the left.

Definition 6.5. *Keeping the notation above, we say a G-space X is free if $g \cdot x = x$ implies that $g = e$.*

Recall that for a "nice" topological group G, there is a universal G-bundle:

$$G \to EG \to BG$$

where EG is a contractible free (right) G-space.

Definition 6.6. *Given a (left) G-space X, one can perform the Borel construction:*

$$EG \times_G X = EG \times X / \sim$$

where $\sim$ is the equivalence relation generated by $(tg, x) \sim (t, gx)$ for all $t \in EG, x \in X, g \in G$.

Remark 6.7. *It is easy to see from this definition, that $EG \times_G G = EG \simeq *$. Here G is viewed as a left G-space via left multiplication on itself and we are using the standard notation of $*$ to denote a space which consists of a single point.*

The next lemma collects some well-known elementary facts about the Borel construction:

Lemma 6.8. *Let G be a topological group and X be a G-space then*

$$\pi_1 : EG \times_G X \to BG = EG/G,$$

the map induced by projection onto the first factor, is a bundle map with fiber X.

Furthermore, if X is a free G-space, then the map

$$\pi_2 : EG \times_G X \to X/G,$$

induced by projection onto the second factor is a bundle map with fiber EG. Since EG is contractible, this implies π_2 is a (weak) homotopy equivalence.

Let G_1, G_2 be two topological groups and let $\mu : G_1 \to G_2$ be a homomorphism of topological groups. Suppose further that you are given a μ-equivariant map

$$f : X_1 \to X_2,$$

where X_i is a left G_i-space for $i = 1, 2$. In other words, f satisfies:

$$f(g \cdot x) = \mu(g) \cdot f(x)$$

for all $g \in G_1, x \in X_1$.

Furthermore recall, from the functoriality of the construction of the universal principal G-bundle, one has the map $E(\mu) : EG_1 \to EG_2$ which is also μ-equivariant which induces the map $B(\mu) : BG_1 \to BG_2$ on the level of the quotient spaces.

Thus one can look at

$$E(\mu) \times f : EG_1 \times X_1 \to EG_2 \times X_2.$$

It is easy to check that this map respects the equivalence relations used in forming the respective Borel constructions and thus one gets a map:

$$E(\mu) \bar{\times} f : EG_1 \times_{G_1} X_1 \to EG_2 \times_{G_2} X_2.$$

There is a commutative diagram in this context that is very useful. We state it as the next lemma.

Lemma 6.9. *Given $\mu : G_1 \to G_2$ a homomorphism of topological groups and $f : X_1 \to X_2$ a μ-equivariant map one has the following commutative diagram:*

$$\begin{array}{ccccc} X_1 & \longrightarrow & EG_1 \times_{G_1} X_1 & \longrightarrow & BG_1 \\ \Big\downarrow f & & \Big\downarrow E(\mu)\bar{\times} f & & \Big\downarrow B(\mu) \\ X_2 & \longrightarrow & EG_2 \times_{G_2} X_2 & \longrightarrow & BG_2 \end{array}$$

where each row is a fiber bundle. So in particular, if $B(\mu)$ and f are weak homotopy equivalences, then so is $E(\mu)\bar{\times} f$.

Proof. It is a routine exercise to check that the diagram commutes. The last sentence then follows once again from the five lemma applied to the long exact sequences in homotopy for the two bundles. □

Lemma 6.9 gives us a convenient way to restate part of the results of proposition 6.1.

Proposition 6.10. *For $k \geq 3$ one has that*

$$ESO(3) \times_{SO(3)} F(S^2, k) = K(PB_{k-3}(\mathbb{R}^2 - Q_2), 1).$$

Here we are using the natural action of $SO(3)$ on $F(S^2, k)$ described before. (We are using the convention, that $PB_0(X)$ denotes the trivial group.)

Proof. Fix $k \geq 3$, then from proposition 6.1, one has a $SO(3)$-equivariant homotopy equivalence

$$\lambda : SO(3) \times Y \to F(S^2, k).$$

where we have denoted $F(S^2 - Q_3, k - 3)$ by Y for convenience. (This means that Y is a point when $k = 3$.)

Applying lemma 6.9 using $Id : SO(3) \to SO(3)$ as μ, and λ as the equivariant map, one sees immediately that since $B(\mu)$ and λ are homotopy equivalences, then

$$E(Id)\bar{\times}\lambda : ESO(3) \times_{SO(3)} (SO(3) \times Y) \to ESO(3) \times_{SO(3)} F(S^2, k)$$

is a (weak) homotopy equivalence. However,

$$\begin{aligned} ESO(3) \times_{SO(3)} (SO(3) \times Y) &= (ESO(3) \times_{SO(3)} SO(3)) \times Y \\ &= ESO(3) \times Y \\ &\simeq Y \end{aligned}$$

by remark 6.7 and the contractibility of $ESO(3)$. Theorem 2.7 then gives us the desired result. □

Now notice the following. If X is a left G-space, then $F(X, k)$ is also a G-space via a diagonal action (as any group action takes distinct points to distinct points). On the other hand Σ_k acts on $F(X, k)$ on the right by permuting coordinates. It is easy to see that the Σ_k-action on $F(X, k)$ commutes with the G-action. Thus if we look at the Σ_k action on $EG \times F(X, k)$ where Σ_k acts purely on the right factor by permuting coordinates, then it is easy to check that this action descends to an action of Σ_k on the Borel construction $EG \times_G F(X, k)$. It is also routine to check that this action of Σ_k on the Borel contruction is still free and that

$$(EG \times_G F(X, k))/\Sigma_k = EG \times_G SF(X, k)$$

where recall that $SF(X, k)$ denotes $F(X, k)/\Sigma_k$.

We summarize this useful fact in the following remark:

Remark 6.11. *Let G be a topological group and let X be a left G-space then there is a natural Σ_k-covering map:*

$$EG \times_G F(X,k) \xrightarrow{\pi} EG \times_G SF(X,k)$$

where the natural diagonal actions of G on $F(X,k)$ and $SF(X,k)$ are used to form the Borel constructions.

Remark 6.11 and proposition 6.10 have the following immediate corollary:

Corollary 6.12. *$ESO(3) \times_{SO(3)} SF(S^2,k)$ is a $K(\pi,1)$ space for all $k \geq 3$.*

One bad thing about the Borel constructions so far, is that although they produced $K(\pi,1)$ spaces, these spaces did not have $PB_k(S^2)$ as their fundamental group. We will now fix this point.

If a topological group G has a universal cover $\tilde{G}$ (which we always assume is connected), then it is elementary to show that $\tilde{G}$ has the structure of a topological group such that the covering map $\mu : \tilde{G} \to G$ is a homomorphism of topological groups. $\tilde{G}$ is called the covering group of G.

Now given a (left) G-space X, X can also be viewed as a $\tilde{G}$-space via μ. Thus the identity map of X is μ-equivariant in our previous terminology. It then follows from lemma 6.9 that we have the following commutative diagram where each row is a fibration:

$$\begin{array}{ccccc}
X & \longrightarrow & E\tilde{G} \times_{\tilde{G}} X & \longrightarrow & B\tilde{G} \\
\Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle E(\mu)\bar{\times} Id} & & \Big\downarrow{\scriptstyle B(\mu)} \\
X & \longrightarrow & EG \times_G X & \longrightarrow & BG
\end{array}$$

We wish to study the vertical map in the middle. To do this, we need to look at the vertical maps on the sides first. It is obvious that the identity map is a homotopy equivalence, so let us first look at $B(\mu)$. Recall the following elementary remarks:

Remark 6.13. *For any topological group G, BG is always path connected and $\pi_n(BG) \cong \pi_{n-1}(G)$ for $n > 1$.*

Proof. We have the universal principal G-bundle $EG \to BG$. The result follows immediately from the long exact sequence in homotopy for this fibration and the contractibility of EG. □

Remark 6.14. *A covering map $\pi : \tilde{X} \to X$ induces isomorphisms between $\pi_n(\tilde{X})$ and $\pi_n(X)$ for all $n > 1$.*

It follows from these remarks that $\mu : \tilde{G} \to G$ induces isomorphisms between the higher homotopy groups (π_n for $n > 1$) and hence that $B(\mu)$ induces isomorphisms in π_n for all $n \neq 2$. (Here recall that we insist that $\tilde{G}$ and hence also G is path connected.) Of course it induces the zero map on the π_2 level as

$$\pi_2(B\tilde{G}) = \pi_1(\tilde{G}) = 0.$$

With this knowledge of $B(\mu)$ and by using the usual argument of applying the five-lemma to the vertical maps between the long exact sequences in homotopy for the two fibrations, one finds easily that $E(\mu)\bar{\times} Id$ induces an isomorphism in π_n for all $n \neq 1, 2$. Using the refined form of the five-lemma (stated in lemma 4.22), it also follows easily that $E(\mu)\bar{\times} Id$ induces a monomorphism on the π_2 level.

This is the main step in proving the following useful lemma:

Lemma 6.15. *Let G be a path connected, topological group. Let $\tilde{G}$ be the universal covering group with $\mu : \tilde{G} \to G$ the covering map. Furthermore suppose X is a G-space. Then $E(\mu)\bar{\times} Id$ induces isomorphisms*

$$\pi_n(E\tilde{G} \times_{\tilde{G}} X) \cong \pi_n(EG \times_G X)$$

for all $n \neq 1, 2$ and a monomorphism

$$\pi_2(E\tilde{G} \times_{\tilde{G}} X) \overset{1-1}{\to} \pi_2(EG \times_G X).$$

Furthermore,

$$\pi_1(E\tilde{G} \times_{\tilde{G}} X) \cong \pi_1(X)$$

Proof. We have already proved everything besides the final isomorphism listed. To prove this, look at the fiber bundle

$$X \to E\tilde{G} \times_{\tilde{G}} X \to B\tilde{G}.$$

Now we saw, in the paragraph preceding the lemma, that $\pi_2(B\tilde{G}) = 0$, but one also has

$$\pi_1(B\tilde{G}) \cong \pi_0(\tilde{G}) = 0.$$

Using this, it is easy to obtain our desired result from the long exact sequence in homotopy for the fiber bundle above. $\square$

Now it is a well known fact that $\widetilde{SO(3)}$ is S^3, the group of unit quarternions, which is topologically a 3-sphere. Thus applying lemma 6.15 to this group and using proposition 6.10 and corollary 6.12, one easily obtains:

Corollary 6.16.

$$ES^3 \times_{S^3} F(S^2, k) = K(PB_k(S^2), 1)$$

and

$$ES^3 \times_{S^3} SF(S^2, k) = K(B_k(S^2), 1),$$

for $k \geq 3$.

Thus, once again we see that the configuration space provides a good model (this time using a Borel construction) for the (pure) braid group of the underlying space.

Thus the only surface whose configuration space we have not studied yet is $\mathbb{R}P^2$. We will do this now, in the next section.

7. Configuration spaces for the real projective plane

7.1. **Orbit configuration spaces.** In our study of the configuration spaces of $\mathbb{R}P^2$, we will find the concept of an orbit configuration space quite useful. These were introduced by M. Xicoténcatl and are defined as follows:

Definition 7.1. *Let G be a group and X a G-space. Then*

$$F_G(X, k) = \{(x_1, \ldots, x_k) \in X^k | x_i, x_j \text{ are in different } G \text{ orbits if } i \neq j\}.$$

$F_G(X, k)$ is called the k-fold orbit configuration space of X.

Remark 7.2. *Thus we see that $F_G(X,k)$ is the space of k-tuples of points which lie in distinct G orbits of X. Note that $F_G(X,1) = X$ and $F_1(X,k) = F(X,k)$ for the action of the trivial group 1 on X.*

We will now prove some elementary but useful properties of these orbit configuration spaces. First some necessary definitions.

Definition 7.3. *Given two groups G_1 and G_2, a left G_1-space X is said to have a compatible G_2 action if*
(a) X is a left G_2-space.
(b) For every $g_1 \in G_1, g_2 \in G_2, x \in X$, there exists $g_1' \in G_1$ such that

$$g_2 \cdot (g_1 \cdot x) = g_1' \cdot (g_2 \cdot x).$$

If it is always possible to take $g_1' = g_1$, we say that the two actions commute.

Remark 7.4. *Notice that if a G_1-space X has a compatible G_2 action then the G_2 action takes G_1-orbits to G_1-orbits and hence induces a G_2 action on X/G_1. (To guarantee the continuity of this action, one should assume that G_2 is locally compact, Hausdorff so that $G_2 \times X \to G_2 \times X/G_1$ is a quotient map. This holds for all Lie groups and hence discrete groups.)*

Remark 7.5. *The most important examples of compatible actions are:*
(a) commuting actions.
(b) When $G_1 = G_2 = G$ and both groups act in the same way on X. These are compatible actions as

$$g_2 \cdot (g_1 \cdot x) = (g_2 g_1 g_2^{-1}) \cdot (g_2 \cdot x)$$

and so we can take $g_1' = g_2 g_1 g_2^{-1}$.

Lemma 7.6. *If X is a G_1-space which supports a compatible G_2-action, then there is a natural G_2^k action on $F_{G_1}(X,k)$ defined via*

$$(g_1, \ldots, g_k) \cdot (x_1, \ldots, x_k) = (g_1 \cdot x_1, \ldots, g_k \cdot x_k)$$

for all $(g_1, \ldots, g_k) \in G_2^k$ and $(x_1, \ldots, x_k) \in F_{G_1}(X,k)$. Notice of course, this also gives a G_2 action on $F_{G_1}(X,k)$ obtained by precomposing the above action with the diagonal homomorphism $\Delta : G_2 \to G_2^k$ which sends g to $(g, \cdots, g)$.

Proof. The proof follows easily from remark 7.4 and is left to the reader. □

¿From lemma 7.6, we see that there is always a natural action of G^k on $F_G(X,k)$. If the original G-action on X is reasonable, we can say something about this G^k-action on $F_G(X,k)$:

Proposition 7.7. *If $\pi : X \to X/G$ is a principal G-bundle, there is a map*

$$\bar{\pi} : F_G(X,k) \to F(X/G,k)$$

which is a principal G^k-bundle, where

$$\bar{\pi}(x_1, \ldots, x_k) = (\pi(x_1), \ldots, \pi(x_k))$$

and the G^k action on $F_G(X,k)$ is that which is described in lemma 7.6.

Proof. First notice that by taking a k-fold product we get a principal G^k-bundle

$$G^k \to X^k \stackrel{\times\pi}{\to} (X/G)^k.$$

Then we can take the pullback of this bundle with respect to the inclusion map $i : F(X/G, k) \to (X/G)^k$ and hence get the commutative diagram:

$$\begin{array}{ccc} G^k & \xrightarrow[Id]{} & G^k \\ \downarrow & & \downarrow \\ E & \longrightarrow & X^k \\ \downarrow{\scriptstyle\bar{\pi}} & & \downarrow{\scriptstyle\times\pi} \\ F(X/G,k) & \xrightarrow[i]{} & (X/G)^k \end{array}$$

Thus the left hand column is also a principal G^k-bundle and the reader can easily check from the definition of a pullback (see [Bre]), that E is naturally identified with $F_G(X, k)$ and that with this identification, $\bar{\pi}$ has the form stated above and the G^k action on $E = F_G(X, k)$ is that which is described in lemma 7.6. □

Remark 7.8. *Recall, that when G is discrete, a principal G-bundle is the same thing as a regular cover with covering group G. With this, it follows easily from proposition 7.7, that if G is a finite group and X a free G-space, then there is a covering map*

$$\pi : F_G(X, k) \to F(X/G, k)$$

with covering group G^k.

Proposition 7.7 shows us immediately how orbit configuration spaces can help us understand the configuration space of $\mathbb{R}P^2$. To see this, we can take S^2 as a $\mathbb{Z}/2\mathbb{Z}$-space where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ is acting via the antipodal map. Then proposition 7.7 gives us the following covering map

$$(\mathbb{Z}/2\mathbb{Z})^k \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k) \stackrel{\pi}{\to} F(\mathbb{R}P^2, k).$$

Thus we need to study $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k)$. To do this, we will need to use the analogue of the Fadell-Neuwirth theorem (theorem 2.5) for orbit configuration spaces. We will state and prove this analogue next.

Definition 7.9. *Given a free G-space X, we will use the notation* O_k *to denote the disjoint union of k distinct G-orbits. Notice* $X - O_k$ *is hence also a G-space.*

Theorem 7.10. *Let G be a finite group and X be a free G-space. Suppose further that X is a connected manifold without boundary with dimension* ≥ 1. *Then for all* $n > k$, *there are fibrations*

$$F_G(X - O_k, n - k) \to F_G(X, n) \stackrel{\pi}{\to} F_G(X, k)$$

where π *is projection onto the first k factors.*

Proof. Once one has verified that π is a fibration, it is an easy exercise to see that the fiber is as described in the statement of this theorem. Thus, we will only show that π is a fibration.

Since the composition of fibrations is a fibration, it is enough to show that π is a fibration in the case when $n = k+1$, since an arbitrary projection from $F_G(X, n)$ to

$F_G(X,k)$ can be broken up as a composition of maps of this sort. Thus, we assume that $n = k+1$ from now on.

Let the order of G be s and write $G = \{g_1, \ldots, g_s\}$. Define $\lambda : F_G(X,k) \to F(X, sk)$ by

$$\lambda(x_1, \ldots, x_k) = (g_1x_1, g_2x_1, \ldots, g_sx_1, g_1x_2, \ldots, g_sx_2, \ldots, g_1x_k, \ldots, g_sx_k).$$

The following diagram commutes:

$$\begin{tikzcd}
F(M - Q_{ks}, 1) \arrow[r] & F(M, ks+1) \arrow[r, "\mu"] & F(M, ks) \\
F_G(M - O_k, 1) \arrow[r] \arrow[u, "Id"] & E \arrow[r, "\bar{\mu}"] \arrow[u, "\bar{\lambda}"] & F_G(M, k) \arrow[u, "\lambda"]
\end{tikzcd}$$

Here the top row is a fibration by theorem 2.5, where μ is projection onto the first ks factors. The bottom row is the pullback of this fibration under the map λ.

Thus $\bar{\mu}$ is a fibration. It remains only to show that the pullback E is naturally homeomorphic to $F_G(M, k+1)$. Recall that

$$E = \{(x,y) \in F_G(M,k) \times F(M, ks+1) | \lambda(x) = \mu(y)\}.$$

Define $\theta : F_G(M, k+1) \to F_G(M,k) \times F(M, ks+1)$ by

$$\theta(x_1, \ldots, x_{k+1}) = ((x_1, \ldots, x_k), (g_1x_1, \ldots, g_sx_1, \ldots, g_1x_k, \ldots, g_sx_k, x_{k+1})).$$

It is easy to check that in fact $\theta : F_G(M, k+1) \to E$.

Now, $\theta : F_G(M, k+1) \to E$ has an inverse given by

$$((x_1, \ldots, x_k), (g_1x_1, \ldots, g_sx_k, x_{k+1})) \to (x_1, \ldots, x_k, x_{k+1}).$$

Thus, θ is our desired homeomorphism and furthermore $\bar{\mu} \circ \theta : F_G(X, k+1) \to F_G(X,k)$ is indeed projection onto the first k coordinates. This is a fibration as $\bar{\mu}$ is. Thus the theorem is proven. □

(The authors would like to thank Dan Cohen for informing us about the nice proof above. Previous proofs consisted of going through the proof of theorem 2.5 carefully and doing the necessary modifications, which is considerably more messy.)

7.2. **Application of orbit configuration spaces to $F(\mathbb{R}P^2, k)$.** First, we need to study $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k)$.

Lemma 7.11. *Let S^2 be given a $\mathbb{Z}/2\mathbb{Z}$ action via the antipodal map.*

Then $F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k)$ is a $K(\pi, 1)$ space for all $n, k \geq 1$.

Furthermore $\pi_1(F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k))$ is a polyfree group of cohomological dimension k and it has a polyfree series with exponents

$$(2(n+k-1)-1, \ldots, 2(n+1)-1, 2(n+0)-1).$$

Proof. We will prove this lemma by induction on k. First the case when $k = 1$. Here we have that

$$F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, 1) = S^2 - O_n = \mathbb{R}^2 - Q_{2n-1} = K(F_{2n-1}, 1).$$

Thus the result follows as F_{2n-1}, the free group on $2n-1$ generators, is polyfree, of cohomological dimension one with the stated exponent.

So without loss of generality, $k > 1$ and we can assume the theorem is proven for numbers smaller than k.

Now by theorem 7.1, we have a fibration

$$F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_{n+k-1}, 1) \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k) \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k-1).$$

By induction, the fiber and the base space are $K(\pi, 1)$ spaces and hence, by the long exact sequence in homotopy, so is the total space.

Furthermore, on the level of fundamental groups, the long exact sequence in homotopy for the fibration above gives us a short exact sequence of groups:

$$\pi_1(F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_{n+k-1}, 1)) \to \pi_1(F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k)) \to \pi_1(F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k-1)).$$

By induction, the base (quotient) group is polyfree, of cohomological dimension $k-1$, with exponents:

$$(2(n+k-2)-1, \ldots, 2(n+0)-1)$$

and also the fiber group (kernel) is a free group of rank $2(n+k-1)-1$. ¿From this, the properties of $\pi_1(F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_n, k))$ stated in the lemma, follow easily. □

Unfortunately, lemma 7.11 does not tell us anything about $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k)$ directly - the lemma only applies if at least one $\mathbb{Z}/2\mathbb{Z}$-orbit has been removed from S^2. We have to do some genuine geometric analysis to go any further, so let us analyze $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ and show it is homotopy equivalent to $SO(3)$ in a nice way.

First, let us notice that if we view S^2 as the unit vectors in $\mathbb{R}^3$, then $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is naturally identified as the space of pairs of linearly independent unit vectors of $\mathbb{R}^3$.

Thus, inside of $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is the subspace

$$U = \{(x, y) | x \text{ is orthogonal to } y \text{ and } x, y \in S^2\}$$

of pairs of orthogonal unit vectors. Notice that U can be naturally identified with the unit tangent bundle of S^2 (We will not need this fact).

$SO(3)$ acts naturally on S^2 and this action commutes with the antipodal map, thus by lemma 7.6, $SO(3)$ acts diagonally on $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$.

It is easy to see that this action preserves the subspace U. In fact, even more is true. Let e_i be the ith standard unit vector of $\mathbb{R}^3$ for $i = 1, 2, 3$. Then given $(x, y) \in U$, since there are elements of $O(3)$ mapping (e_1, e_2, e_3) to any other given orthonormal basis, it is easy to see that there is an element of $SO(3)$ mapping (e_1, e_2) to (x, y). In fact this element is unique as any element of $SO(3)$ mapping (e_1, e_2) to (x, y) would have to map $e_1 \times e_2 = e_3$ to $x \times y$ and so is determined on a basis. (Here $\times$ stands for the cross product in $\mathbb{R}^3$.)

Thus we see that the $SO(3)$ action on U is transitive and free and hence $U = SO(3)$. Furthermore, the $SO(3)$ action on U corresponds to the left multiplication action on $SO(3)$ under this correspondence.

Proposition 7.12. *The inclusion $i : U \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is a $SO(3)$-equivariant homotopy equivalence. Furthermore $U = SO(3)$, with $SO(3)$-action given by left multiplication.*

Proof. We have already proven the final sentence of the proposition in the preceding paragraph. It remains only to show that i is a homotopy equivalence.

We will need a bit of notation. For $x, y \in \mathbb{R}^n$, let $x \cdot y$ denote the dot product of x and y. Let $\| x \| = \sqrt{x \cdot x}$ denote the norm of x.

Now we construct $R : F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2) \times I \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ as follows:

$$R(x, y, t) = (x, \frac{y - t(x \cdot y)x}{\| y - t(x \cdot y)x \|}).$$

Notice R is well defined as y is not a multiple of x. The following properties of R follow easily:
(a) $R(x, y, 0) = (x, y)$ for all $(x, y) \in F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$,
(b) $R(x, y, 1) \in U$,
(c) $R(x, y, t) = (x, y)$ for all $(x, y) \in U, t \in I$.
(d) $R(\alpha x, \alpha y, t) = \alpha R(x, y, t)$ for all $(x, y) \in F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2), t \in I, \alpha \in O(3)$. Here $O(3)$ is acting diagonally on $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ in the natural way.

Properties (a)-(c) tell us that U is a strong deformation retract of $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ which is all we need to finish the proof of the proposition. Property (d) says in fact, that there is a strong deformation through $SO(3)$-equivariant maps of $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ to itself.

(Notice, that R, is in effect, performing a Gram-Schmidt orthogonalization process globally.) □

Corollary 7.13. *If $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is given the diagonal $SO(3)$ action induced from the natural action of $SO(3)$ on S^2 then $ESO(3) \times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is (weakly) contractible.*

Proof. Proposition 7.12 shows that $i : U = SO(3) \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ is a $SO(3)$-equivariant homotopy equivalence. By lemma 6.9, it follows that

$$Id \bar{\times} i : ESO(3) \times_{SO(3)} U \to ESO(3) \times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$$

is a (weak) homotopy equivalence.

On the other hand,

$$ESO(3) \times_{SO(3)} U = ESO(3) \times_{SO(3)} SO(3) = ESO(3) \simeq *$$

□

This was the crucial ingredient to the following proposition:

Proposition 7.14. *If $F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k)$ is given the diagonal $SO(3)$-action induced from the natural $SO(3)$ action on S^2 then $ESO(3) \times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k)$ is a $K(\pi, 1)$-space for all $k \geq 2$.*

Proof. We will prove this by induction on k. The case $k = 2$ was proven in corollary 7.13 so assume $k > 2$ and that we have proven the proposition for smaller numbers.

Let $\pi : F_{\mathbb{Z}/2\mathbb{Z}}(S^2, k) \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)$ be projection onto the first 2 factors. Notice that π is $SO(3)$-equivariant.

By theorem 7.1, π is a fibration with fiber $F_{\mathbb{Z}/2\mathbb{Z}}(S^2 - O_2, k-2)$ which is a $K(\pi, 1)$-space by lemma 7.11. Thus π induces an isomorphism in π_i for $i \neq 1, 2$. However $\pi_2(F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2)) = \pi_2(SO(3)) = 0$ so in fact in the long exact sequence for the fibration above, the boundary map from $\pi_2(F_{\mathbb{Z}/2\mathbb{Z}}(S^2, 2))$ to $\pi_1(fiber)$ necessarily vanishes and π is an isomorphism on the π_2 level also.

By lemma 6.9, we have the following commutative diagram

$$\begin{array}{ccccc} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k) & \longrightarrow & ESO(3)\times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k) & \longrightarrow & BSO(3) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle Id\bar{\times}\pi} & & \Big\downarrow{\scriptstyle Id} \\ F_{\mathbb{Z}/2\mathbb{Z}}(S^2,2) & \longrightarrow & ESO(3)\times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,2) & \longrightarrow & BSO(3) \end{array}$$

where each row is a fibration. In the preceding paragraph, we saw that π induced isomorphisms in π_i for all $i \neq 1$. Thus by the refined form of the five lemma given in lemma 4.22, we see that $Id\bar{\times}\pi$ induces a monomorphism in π_i for all $i \neq 1$. Since $\pi_i(ESO(3)\times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,2)) = 0$ for all i this lets us conclude that $\pi_i(ESO(3)\times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k)) = 0$ for all $i \neq 1$ which is what we set out to prove. □

Now we are finally ready to study the configuration space $F(\mathbb{R}P^2,k)$. Recall we had the covering map:

$$(\mathbb{Z}/2\mathbb{Z})^k \to F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k) \xrightarrow{\pi} F(\mathbb{R}P^2,k).$$

As before, $SO(3)$ acts diagonally on $F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k)$ using the natural action of $SO(3)$ on S^2. Of course, $SO(3)$ also acts naturally on $\mathbb{R}P^2$ viewed as the space of lines in $\mathbb{R}^3$. Hence, $SO(3)$ acts diagonally on $F(\mathbb{R}P^2,k)$. Furthermore, it is easy to see, that the covering map π above is $SO(3)$-equivariant with respect to these actions. (For $k=1$, π is just the map taking a unit vector, to the line it spans.)

Being a covering map, π induces isomorphisms in π_i for all $i \neq 1$, and a monomorphism in π_1. We have the usual commutative diagram

$$\begin{array}{ccccc} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k) & \longrightarrow & ESO(3)\times_{SO(3)} F_{\mathbb{Z}/2\mathbb{Z}}(S^2,k) & \longrightarrow & BSO(3) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle Id\bar{\times}\pi} & & \Big\downarrow{\scriptstyle Id} \\ F(\mathbb{R}P^2,k) & \longrightarrow & ESO(3)\times_{SO(3)} F(\mathbb{R}P^2,k) & \longrightarrow & BSO(3) \end{array}$$

where each row is a fibration.

¿From the five lemma (lemma 4.22), it follows easily that $Id\bar{\times}\pi$ induces an epimorphism in π_i for $i \neq 1$. Thus from proposition 7.14, it follows that $ESO(3)\times_{SO(3)} F(\mathbb{R}P^2,k)$ is a $K(\pi,1)$-space. This is the result we wanted in this section and we state it as:

Theorem 7.15. *$ESO(3)\times_{SO(3)} F(\mathbb{R}P^2,k)$ is a $K(\pi,1)$-space if $k \geq 2$ and furthermore*

$$ES^3 \times_{S^3} F(\mathbb{R}P^2,k) = K(PB_k(\mathbb{R}P^2),1)$$

and

$$ES^3 \times_{S^3} SF(\mathbb{R}P^2,k) = K(B_k(\mathbb{R}P^2),1).$$

Here, the $SO(3)$ action is induced from the natural action of $SO(3)$ on $\mathbb{R}P^2$ viewed as the space of lines in $\mathbb{R}^3$. The S^3 action is obtained from the $SO(3)$ action using that S^3 is the universal covering group of $SO(3)$.

Proof. Most of the theorem was proven in the preceding paragraph. The only thing remaining is the statement about the S^3 Borel constructions which follows immediately from lemma 6.15 and remark 6.11. □

Remark 7.16. *¿From the fibration,*

$$F(\mathbb{R}P^2, k) \to ES^3 \times_{S^3} F(\mathbb{R}P^2, k) \to BS^3$$

and the fact that $ES^3 \times_{S^3} F(\mathbb{R}P^2, k)$ is a $K(\pi, 1)$-space, we see easily that

$$\pi_n(F(\mathbb{R}P^2, k)) \cong \pi_{n+1}(BS^3) \cong \pi_n(S^3)$$

for $n \geq 2$.

Thus, $F(\mathbb{R}P^2, k)$ has the same higher homotopy as the 3-sphere.

This completes our analysis of $F(M, k)$ for 2-manifolds M without boundary. The conclusion is that these are always $K(PB_k(M), 1)$-spaces except when M is S^2 or $\mathbb{R}P^2$. However in these cases, there is an associated Borel construction which is a $K(PB_k(M), 1)$.

For now, these results might seem to be a pretty formal analysis of the homotopy type of the configuration spaces of surfaces - however we will see their true power when we begin talking about labeled configuration spaces, and the "classical" connection of these labeled configuration spaces with loop spaces.

Furthermore, Borel constructions will provide a crucial tool in connecting the braid groups we have studied, to mapping class groups, another important class of groups associated to surfaces, which we will introduce and study later on in these notes.

8. Mapping class groups

Let M denote an orientable surface and let $Top(M)$ be the group of orientation preserving homeomorphisms of M. We make it a topological group by giving it the compact open topology.

Let Q_k be a set of k distinct points in M and let $Top(M, Q_k)$ be the topological subgroup of $Top(M)$ which leaves the set Q_k invariant.

Recall the standard isotopy lemma, (see [**?**])

Lemma 8.1 (Isotopy Lemma)**.** *Let M be a connected smooth manifold of dimension strictly bigger than one. Then for $(x_1, \dots, x_k), (y_1, \dots, y_k) \in F(M, k)$, there exists a diffeomorphism $\phi : M \to M$ such that ϕ is isotopic to the identity and $\phi(x_i) = (y_i)$ for $1 \leq i \leq k$.*

¿From the isotopy lemma, there is an orientation preserving homeomorphism of M carrying one set of k distinct points Q_k to any other set of k distinct points Q'_k, and thus $Top(M, Q_k)$ is conjugate to $Top(M, Q'_k)$ in $Top(M)$ and so we will sometimes write $Top(M, k)$ when the points are understood.

We will also look at $PTop(M, Q_k)$, the topological subgroup of $Top(M, Q_k)$ consisting of homeomorphisms which actually fixe the points of Q_k pointwise. Thus $PTop(M, Q_k)$ is the kernel of the natural homomorphism from $Top(M, Q_k) \to \Sigma_k$ induced by sending a homeomorphism leaving the set Q_k invariant to the permutation it induces on those points. Again, we will write $PTop(M, k)$ when the points Q_k are understood.

Recall that if M is a surface then the inclusion of the group of orientation preserving diffeomorphisms $Diff^+(M) \to \ Top(M)$ is a homotopy equivalence. Some of the motivation for considering $PTop(M, k)$ is described next.

Properties of these subgroups frequently correspond to a lifting question relating branched covering spaces. Namely given a branched cover $N \to M$ which is branched over a finite set Q_k, information about $Top(M,k)$ gives information about $Top(N)$. That is, homeomorphisms of M which leave the branch set invariant lift (in possibly more than one way) to homeomorphisms of N.

One direction of these notes is to obtain information concerning $Top(M,k)$ in order to obtain information about $Top(N)$. This approach is carried out in a few cases in these notes for the specific examples of surfaces of genus 0,1, and 2. This approach dates back to Hurwitz, and has been exploited in [BH], [**?**]; these classical methods have proven to be useful also for cohomological analysis.

In a few of the applications, pointed versions of these constructions are useful. For example, the cohomology of the "pointed mapping class group" for a genus 1 surface with marked points (to be defined below) has a clean cohomological description. The analogous description without the assumption of "pointed maps" has a technically more complicated description. This structure is analogous to the behavior of certain function spaces. That is, the space of pointed maps from a circle to a space X, the loop space ΩX is frequently accessible from a homological point of view. At the same time, the space of free maps from a circle to X, the free loop space ΛX, is frequently more delicate.

Mapping class groups, in some cases, reflect similar behavior as we will see later when we study the pointed mapping class group for punctured copies of genus zero, one, and two surfaces.

Recall given a topological group G, the path components $pi_0(G)$ form a (discrete) group where the group structure is induced from G.

Definition 8.2. *Let M be a closed orientable surface of genus g. The mapping class group $\Gamma_g^k = \pi_0(Top(M, Q_k))$.*

The pure mapping class group $P\Gamma_g^k$ is the kernel of the natural homomorphism $\Gamma_g^k \to \Sigma_k$, induces by sending a homeomorphism to the permutation it induces a the points Q_k.

Definition 8.3. *Let M be a closed orientable surface of genus g, Q_k a set of k distinct points of M and p a fixed point in $M - Q_k$.*

The pointed mapping class group $\Gamma_g^{k,1}$ is the group of path components of the orientation preserving homeomorphisms which (1) preserve the point p, and (2) leave the set Q_k invariant.

The pure pointed mapping class group $P\Gamma_g^{k,1}$ is the kernel of the natural homomorphism $\Gamma_g^{k,1} \to \Sigma_k$.

The group $Top(M)$ acts on the configuration space of points in M, $F(M,k)$, diagonally. A "folk theorem" that has been useful gives (1) there are natural $K(\pi,1)'s$ obtained from the associated Borel construction (homotopy orbit spaces) for groups acting on configuration spaces, and
(2) these configuration spaces are analogous to homogeneous spaces in the sense

that they are frequently homeomorphic to a quotient of a topological group by a closed subgroup.

Namely, let G be a subgroup of $Top(M)$, and consider the diagonal action of G on $F(M,k)$ together with the homotopy orbit spaces

$$EG \times_G F(M,k)$$

and

$$EG \times_G F(M,k)/\Sigma_k.$$

Lemma 8.4. *Assume that M is a nonempty manifold of dimension at least 1. Then $Top(M,k)$ and $PTop(M,k)$ are closed subgroups of $Top(M)$.*

Proof. Given any point f in the complement of $Top(M,k)$ in $Top(M)$, there is at least one $m_i \in Q_k$ that is taken to a point outside Q_k. The open set of continuous functions that carry m_i to $M - Q_k$ in the complement of $Top(M,k)$ in $Top(M)$ is a neighbourhood of f. Thus it follows that the complement of $Top(M,k)$ in $Top(M)$ is open and the lemma follows for $Top(M,k)$. The proof for $PTop(M,k)$ is similar. □

Consider the natural evaluation map $Top(M) \to F(M,k)$ given by $Top(M)$ acting on the point $(m_1, m_2, ..., m_k)$. Since the isotropy group of the $Top(M)$ action on $F(M,k)$ at $(m_1, \ldots, m_k)$ is the group $Top(M,k)$, there is an induced map

$$\rho : Top(M)/Top(M,k) \to F(M,k).$$

By the isotopy lemma, ρ is onto and hence a continuous bijection.

In the next theorem, we will show among other things, that in fact the map ρ above is a homeomorphism. Thus configuration spaces of surfaces are "homogeneous spaces" of suitable topological groups. (The quotation marks about the word homogeneous space is due to the fact that $Top(M)$ is only a topological group, not a Lie group).

The theorem will be proven from a construction in Steenrod's book "The topology of fibre bundles". Namely, consider

$$H \to G \to G/H$$

where H is a closed subgroup of G, and the map $G \to G/H$ admits "local cross-sections". Then the induced map $BH \to BG$ is the projection map in a fibre bundle with fibre given by the space of left cosets G/H. [Steenrod, page 30]

The definition of "local cross-sections" is given as follows: Let H be a closed subgroup of G with natural quotient map $p : G \to G/H$ and let $x \in G/H$. A local cross-section of $p : G \to G/H$ at x is a continuous function $f : V \to G$ where V is an open neighborhood of x in G/H satisfying $pf(x) = x$ for all $x \in V$. [Steenrod, page 30].

Theorem 8.5. *Assume that M is an orientable surface without boundary. Then*

(1) *There is a principal fibration*

$$Top(M,k) \to Top(M) \to Top(M)/Top(M,k).$$

(2) *The map*

$$\rho : Top(M)/Top(M,k) \to \ F(M,k)$$

is a homeomorphism.

(3) *The homotopy theoretic fibre of the natural map $BTop(M,k) \to \ BTop(M)$ is $F(M,k)$, and $ETop(M) \times_{Top(M)} F(M,k)$ is homotopy equivalent to $BTop(M,k)$.*

The proof of Theorem 8.5 depends on the next lemma. Here, let D^n denote the standard n-disk, i.e., the points in $\mathbb{R}^n$ of euclidean norm at most 1 with interior denoted $D^{o,n}$ with $(0,0,...,0)$ the origin in D^n. The map θ in the next lemma was quite useful in the article by Fadell and Neuwirth [FN] while the formula here was written explicitly in [X].

Lemma 8.6. (1) *There is a continuous map*
$\theta : D^{o,n} \times D^n \to \ D^n$
such that $\theta(x,-)$ fixes the boundary of D^n pointwise, and $\theta(x,x) = (0,0,...,0)$ for every x in $D^{o,n}$.

(2) *If M is a surface without boundary, then there exists a basis of open sets U for the topology of $F(M,k)$ together with local sections*
$\phi : U \to \ Top(M)$ *such that the composite*

$$U \xrightarrow{\phi} Top(M) \to \ Top(M)/Top(M,k) \xrightarrow{\rho} F(M,k)$$

is a homeomorphism onto U.

(3) *The natural map $\rho : Top(M)/Top(M,k) \to \ F(M,k)$ is a homeomorphism.*

Proof. Define $\alpha : D^{o,n} \to \ R^n$ by the formula $\alpha(x) = x/(1-|x|)$, and so $\alpha^{-1}(z) = z/(1+|z|)$.

For a fixed element q in $D^{o,n}$, define $\gamma_q : D^n \to \ D^n$ by the formula

(1) $\gamma_q(y) = y$ for y in the boundary of D^n, and
(2) $\gamma_q(y) = \alpha^{-1}(y/(1-||y||) - q/(1-|q|)$ for y in $D^{o,n}$.

Define $\theta : D^{o,n} \times D^n \to \ D^n$ by the formula $\theta(q,y) = \gamma_q(y)$. Notice that θ is continuous, and $\theta(q,q) = (0,0,...,0)$, and so part (1) of the lemma follows.

To prove part (2), consider a point $(p_1,p_2,...,p_k)$ in $F(M,k)$ together with disjoint open discs $D^{o,2}(p_1), D^{o,2}(p_2), ..., D^{o,2}(p_k)$ where $D^{o,2}(p_i)$ is a disc with center p_i. (There is a choice of homeomorphism in the identification of each open disc with an open coordinate patch of M; this choice will be suppressed here.) Let U be the open set in $F(M,k)$ given by the product $D^{o,2}(p_1) \times D^{o,2}(p_2) \times ... \times D^{o,2}(p_k)$. The sets U give a basis for the topology of $F(M,k)$ which depend on the choice of the discs $D^{o,2}(p_i)$. Define $\phi : U \to \ Top(M)$ by the formula $\phi((y_1,y_2,...,y_k)) = H$ for H in $Top(M)$ where $(y_1,y_2,...,y_k)$ is in $U = D^{o,2}(p_1) \times D^{o,2}(p_2) \times ... \times D^{o,2}(p_k)$, and H is the homeomorphism of M given as follows.

(1) H(x) = x if x is in the complement in the union of the $\coprod_{1 \leq i \leq k} D^{o,2}(p_i)$, and
(2) $H(x) = \theta(p_i,x)$ if x is in $D^{o,2}(p_i)$.

Clearly H is in $Top(M)$. Notice that ϕ is continuous if an only if the adjoint of ϕ, $adj(\phi) : U \times M \to M$ is continuous as all spaces here are locally compact, and Hausdorff. But then continuity follows at once from the first part of the lemma as $adj(\phi)(x, y) = y$ for y in the boundary of any of the discs $D^{o,2}(p_i)$. The second part of the lemma follows.

To finish the third part of the lemma, it must be checked that the natural map $\rho : Top(M)/Top(M, k) \to F(M, k)$ is a homeomorphism. Notice that part 2 of the lemma gives local sections $\phi : U \to Top(M)$. Thus consider the composite $\lambda : U \to Top(M)/Top(M, k)$ given by the composite $p\phi$ where $p : G \to G/H$ is the natural quotient map. Notice that $\lambda : U \to \lambda(U)$ is a continuous bijection, and $\lambda(U) = \phi^{-1}(U)$. Thus $\lambda(U)$ is open, and the map ϕ is open. Thus ρ is open, and hence a homeomorphism. The lemma follows.

□

Next, the proof of Theorem 8.5 is given.

Proof. By Lemmas 8.6, and 8.4, $Top(M, k)$ is a closed subgroup of $Top(M)$, and local sections exist for $Top(M) \to Top(M)/Top(M, k)$. Thus there is a principal fibration $Top(M, k) \to Top(M) \to Top(M)/Top(M, k)$, and the first part of the theorem follows.

Furthermore, the natural evaluation map $Top(M) \to F(M, k)$ factors through the quotient map $Top(M) \to Top(M)/Top(M, k)$. There is the induced map $\rho : Top(M)/Top(M, k) \to F(M, k)$ which, by lemma 1.4 , is a homeomorphism, and part 2 of the theorem follows.

The third statement in the theorem follows from a construction in N. Steenrod's book [S], at the foot of page 30. Namely, consider $H \to G \to G/H$ where H is a closed subgroup of G, and the map $G \to G/H$ admits "local cross-sections", then $BH \to BG$ is the projection map in a fibre bundle with fibre given by the space of left cosets G/H. □

As a consequence of Theorem 8.5, we can describe some of the Borel constructions we have looked at in previous sections as $K(\pi, 1)$ spaces where the group π is given by certain mapping class groups.

Theorem 8.7. (1) *If $q \geq 3$, the spaces $ESO(3)\times_{SO(3)}F(S^2, q)$, and $ESO(3)\times_{SO(3)} F(S^2, q)/\Sigma_q$ are respectively $K(P\Gamma_0^q, 1)$, and $K(\Gamma_0^q, 1)$.*

(2) *If $M = S^1 \times S^1$, and $q \geq 2$, the spaces $ETop(M) \times_{Top(M)} F(M, q)$, and $ETop(M) \times_{Top(M)} F(M, q)/\Sigma_q$ are respectively $K(P\Gamma_1^q, 1)$, and $K(\Gamma_1^q, 1)$. Furthermore, in these cases $ETop(M) \times_{Top(M)} F(M, q)$ is homotopy equivalent to $ESL(2, Z) \times_{SL(2,Z)} F(S^1 \times S^1 - \{(1, 1)\}, q - 1)$ where $SL(2, Z)$ acts on $S^1 \times S^1 - \{(1, 1)\}$ by the formula*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (u^a v^b, u^c v^d) \tag{2}$$

for

$$\begin{pmatrix} u \\ v \end{pmatrix} \tag{3}$$

in $S^1 \times S^1$.

(3) *If M is a surface of genus g without boundary (possibly with punctures), and* $g \geq 2$*, the spaces* $ETop(M) \times_{Top(M)} F(M,q)$ *and* $ETop(M) \times_{Top(M)} F(M,q)/\Sigma_q$ *are respectively* $K(P\Gamma_g^q, 1)$*, and* $K(\Gamma_g^q, 1)$.

(4) *Let M be a closed orientable surface of genus g with x a point of M, and N* $= M - \{x\}$. *The spaces* $ETop(N) \times_{Top(N)} F(N,q)$*, and* $ETop(N) \times_{Top(N)} F(N,q)/\Sigma_q$ *are respectively* $K(P\Gamma_g^{q,1}, 1)$*, and* $K(\Gamma_g^{q,1}, 1)$.

Proof. Notice that Lemma 2 gives that the fundamental group of $ETop(M) \times_{Top(M)} F(M,q)$ is isomorphic to $\pi_0(Top(M,k) = P\Gamma_g^k$. Similarly, the fundamental group of $ETop(M) \times_{Top(M)} F(M,q)/\Sigma_q$ is Γ_g^k The main task is now to determine when $ETop(M) \times_{Top(M)} F(M,q)$ is a $K(\pi, 1)$. If the genus of M is at least 2, then the resulting space is a $K(\pi, 1)$, however, if the genus is 0, or 1, there are some small modifications described below.

A theorem of Smale [Smale] gives that the natural map $SO(3) \rightarrow Diff^+(S^2)$ is a homotopy equivalence. Thus the natural maps $SO(3) \rightarrow Diff^+(S^2) \rightarrow Top(S^2)$ as well as the maps induced on the level of the Borel constructions $ESO(3) \times_{SO(3)} F(S^2,q) \rightarrow ESO(3) \times_{Top(S^2)} F(S^2,q)$ are homotopy equivalences. Since $ESO(3) \times_{SO(3)} F(S^2,q)$ is a $K(\pi, 1$ for $q \geq 3$ (See section ?? in these notes .), the result for part (1) follows from the above lemma.

To prove part(2) of the theorem, notice that a result of Earle, and Eells [EE], notice that there is an exact sequence of groups $1 \rightarrow Diff_0(S^1 \times S^1) \rightarrow Diff^+(S^1 \times S^1) \rightarrow SL(2,Z) \rightarrow 1$ where $Diff_0(S^1 \times S^1)$ is homotopy equivalent to $S^1 \times S^1$ and so $BDiff_0(S^1 \times S^1)$ is homotopy equivalent to $CP^\infty \times CP^\infty$. Furthermore, the map $S^1 \times S^1 \rightarrow Diff_0(S^1 \times S^1)$ given by rotations is a homotopy equivalence by [EE]. Thus the Borel construction $EG \times_G S^1 \times S^1$ is $K(SL(2,Z), 1)$ where G $= Diff^+(S^1 \times S^1)$.

Consider the Borel construction $EG \times_G F(S^1 \times S^1, q)$ together with the projection to $EG \times_G F(S^1 \times S^1, 1)$ having fibre $F(S^1 \times S^1 - Q_1, q-1)$. Recall that $EG \times_G F(S^1 \times S^1, 1)$ is $K(SL(2,Z), 1)$, and observe that the action of $SL(2,Z)$ on $F(S^1 \times S^1 - \{(1,1)\}, q-1)$ is the diagonal natural action of $SL(2,Z)$ on $S^1 \times S^1 - \{(1,1)\}$ where $\{(1,1)\}$ is the identity element in $S^1 \times S^1$ with the following action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = (u^a v^b, u^c v^d). \tag{4}$$

Thus, if $q \geq 2$, the fibre and base of the projection $EG \times_G F(S^1 \times S^1, q) \rightarrow EG \times_G F(S^1 \times S^1, 1)$ are $K(\pi, 1)'s$. Thus $EG \times_G F(S^1 \times S^1, q)$ is also a $K(\pi, 1)$ if $q \geq 2$, and part (3) follows.

To prove part (3), notice that by [EE],each path component of $Diff^+(M)$ is contractible if M is of genus $g \geq 2$. Thus both $BTop(M)$, and $F(M,k)$ are Eilenberg-Mac Lane spaces of type $K(\pi,1)$.

Part(4) is analogous as $BTop(M)$ is a $K(\pi,1)$ where N $= M - \{x\}$. The theorem follows.

□

Recall the definition of the pointed mapping class group: The pointed mapping class group $\Gamma_g^{k,1}$ is the group of path components of the orientation preserving homeomorphisms which (1) preserve the point p in the surface, and (2) leave a set of k other distinct points in M invariant, $\pi_0(Top(M,\{p\},Q_k)$. The pure pointed mapping class group $P\Gamma_g^k$ is the kernel of the natural homomorphism $\Gamma_g^{k,1} \to \Sigma_k$.

Corollary 8.8. (1) *If $q \geq 1$, then the fundamental group of* $ETop(M) \times_{Top(M)} F(M,q+1)/\{1 \times \Sigma_q\}$ *is isomorphic to* $\Gamma_g^{q,1}$.

(2) *If M is of genus zero, and $q \geq 2$, then $ETop(M) \times_{Top(M)} F(M,q+1)/\{1 \times \Sigma_q\}$ is a $K(\Gamma_0^{q,1},1)$.*
(3) *If M is of genus one, and $q \geq 1$, then $ETop(M) \times_{Top(M)} F(M,q+1)/\{1 \times \Sigma_q\}$ is a $K(\Gamma_g^{q,1},1)$.*
Furthermore if $q \geq 1$, then $ETop(M) \times_{Top(M)} F(M,q)$ is homotopy equivalent to $ESL(2,Z) \times_{SL(2,Z)} F(S^1 \times S^1 - \{(1,1)\}, q-1)$ where $SL(2,Z)$ acts on $S^1 \times S^1 - \{(1,1)\}$ by the formula
(4) *If M is of genus at least two, and $q \geq 1$, then $ETop(M) \times_{Top(M)} F(M,q+1)/\{1 \times \Sigma_q\}$ is a $K(\Gamma_g^{q,1},1)$.*

Proof. Consider $ETop(M) \times_{Top(M)} F(M,q+1)/\{1 \times \Sigma_q$ together with the natural projection to $ETop(M) \times_{Top(M)} M$ with fibre $F(M-Q_1,q)/\Sigma_q$. Thus the space $ETop(M) \times_{Top(M)}$ is $BTop(M,1)$ by Theorem 1.2 . Furthermore, the $Top(M,1)$-action on $F(M-Q_1,q)/\Sigma_q$ is induced by the natural diagonal action on $M - Q_1$. Hence the fundamental group is $\Gamma_g^{q,1}$. The hypotheses on the number of points q gives that the resulting spaces are $K(\pi,1)'s$ by Theorem 1.2.

To finish the proof of the corollary, it suffices to notice that $ETop(M) \times_{Top(M)} F(M,q)$ is homotopy equivalent to $ESL(2,Z) \times_{SL(2,Z)} F(S^1 \times S^1 - \{(1,1)\}, q-1)$ where $SL(2,Z)$ acts on $S^1 \times S^1 - \{(1,1)\}$ where M is of genus 1 by the above remarks.

□

9. The Thom construction

9.1. **Basic definitions.** In this section, we will be working in the category of pointed spaces and maps. Recall the basic definitions:

Definition 9.1. *A pointed space is a space X together with a fixed basepoint $* \in X$. A map $f : X \to Y$ between two pointed spaces is said to be a pointed map if $f(*) = *$. A homotopy F between two pointed maps $f, g : X \to Y$ is said to preserve basepoints* *if $F(*\times I) = *$ and we write $f \simeq_* g$. If f and g are homotopic but not necessarily by a homotopy that preserves basepoints, we write $f \simeq g$ and say f is freely homotopic to g.*

We recall some basic constructions. See [Bre] for proofs of the basic properties of these constructions.

Definition 9.2. *Given two pointed spaces X and Y, $X \times Y$ can be made a pointed space by taking $(*, *)$ as basepoint. Define*

$$X \vee Y = \{(x, y) \in X \times Y \text{ such that either } x = * \text{ or } y = *\}.$$

$X \vee Y$ is called the wedge product of X and Y and is a pointed space consisting of the spaces $X = X \times *$ and $Y = * \times Y$ attached at the basepoint $(*, *)$.

Definition 9.3. *Given two pointed spaces X and Y,*

$$X \wedge Y = X \times Y / X \vee Y$$

is called the smash product of X and Y and is itself a pointed space where we set the equivalence class of $X \vee Y$ as the basepoint.

The following lemma collects some useful facts about the wedge and smash products. The proof of these facts is left to the reader and can be found in introductory texts like [Bre] or [Spa].

Lemma 9.4. *Let Top_* denote the category of pointed spaces and maps. We will write $X =_* Y$ if X and Y are isomorphic in this category. Then*
(a) $-\vee-$ and $-\wedge-$ are functors from Top_* to itself, which are covariant in both entries. If $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are two pointed maps we will denote the maps given by these functors as $f \vee g : X_1 \vee Y_1 \to X_2 \vee Y_2$ and $f \wedge g : X_1 \wedge Y_1 \to X_2 \wedge Y_2$.
(b) $- \vee -$ equips the isomorphism classes of objects in Top_* with the structure of a commutative monoid with identity where the identity is the point space $*$. In other words:
(i) $X \vee Y =_* Y \vee X$
(ii) $(X \vee Y) \vee Z =_* X \vee (Y \vee Z)$
(iii) $X \vee * =_* X$.
(c) $- \wedge -$ equips the isomorphism classes of objects in Top_* with a commutative multiplication structure, which together with $\vee$ gives a commutative semiring structure on Top_*. (A semiring is something which satisfies all the axioms of a ring except the existance of additive inverses.) In other words:
(i) $X \wedge Y =_* Y \wedge X$
(ii) $(X \wedge Y) \wedge Z =_* X \wedge (Y \wedge Z)$
(iii) $X \wedge S^0 =_* X$ where $S^0 = \{0, 1\}$ with 0 as the basepoint.
(iv) $(X \vee Y) \wedge Z = (X \wedge Z) \vee (Y \wedge Z)$.
(d) $\vee$ and $\wedge$ are homotopy functors, i.e., if $f \simeq_* f'$ and $g \simeq_* g'$ then we have $f \vee g \simeq_* f' \vee g'$ and $f \wedge g \simeq_* f' \wedge g'$.

Remark 9.5. *The reader should be careful in interpreting the semiring structure in lemma 9.4 since the objects of Top_* do not form a set. However, in practice, this is not a problem as we can always confine ourselves to a suitable set of objects if we wish to use the semiring structure. The reader can check for example, that the "subsemiring" generated by S^0 is naturally identified with the semiring of natural* *numbers. In fact, if we confine ourselves to objects in Top_* where the Euler characteristic χ is defined, and where we can identify $\tilde{H}_*(X \wedge Y)$ as $H_*(X \times Y, X \vee Y)$, then the map taking X to $\chi(X) - 1$ is a semiring morphism to the integers.*

Definition 9.6. *We use $X^{(k)}$ to stand for the kth power of the pointed space X under the smash product multiplication. It is understood then that $X^{(0)} = S^0$ and $X^{(1)} = X$. Notice that, for $k \geq 1$, we have*

$$X^{(k)} = X^k/F$$

where $F = \{(x_1, \ldots, x_k) \in X^k$ such that $x_i = *$ for some $i\}$, is the so called fattened wedge. Notice that the symmetric group Σ_k acts on X^k by permuting coordinates and that this action preserves F. Thus this induces a natural action of Σ_k on $X^{(k)}$ which fixes the basepoint.

Definition 9.7. *Given a space X, the unreduced suspension of X, denoted SX, is the quotient space obtained from $X \times I$ by identifying $X \times \{0\}$ to a point and $X \times \{1\}$ to another point.*
Given a pointed space X, the reduced suspension of X, is defined as

$$\Sigma X = X \times I/(X \times \{0\} \cup * \times I \cup X \times \{1\}).$$

and is given the equivalence class of $* \times I$ as the basepoint.

Remark 9.8. *We will identify the 1-sphere S^1 as the quotient space formed from $[0, 1]$ where we identify 0 and 1 to form a common basepoint. With this, it is easy to see that*

$$S^1 \wedge X =_* \Sigma X.$$

Thus $(S^1)^{(k)} = S^k$ for all $k \geq 0$.

Definition 9.9. *A well pointed space X is a pointed space where the inclusion $* \to X$ is a cofibration. Recall that for such a space, the reduced and unreduced suspensions are homotopy equivalent. (See* [Bre] *or* [Spa]*). Any CW-complex X is well pointed if we take the basepoint to be an element of the zero skeleton $X_{(0)}$.*

9.2. **The Thom construction.** We now introduce an important construction. Let M be a free right Σ_k-space and X be a well pointed space. Then from the final remark in definition 9.6, there is a natural Σ_k action on $X^{(k)}$ which fixes the basepoint. Thus we can form

$$M \times_{\Sigma_k} X^{(k)} = M \times X^{(k)}/\sim,$$

where $(m\sigma, \bar{x}) \sim (m, \sigma\bar{x})$ for all $\sigma \in \Sigma_k$, $\bar{x} \in X^{(k)}$ and $m \in M$.

As in the Borel construction, one can easily show that as the Σ_k action on M is free, the map

$$\pi : M \times_{\Sigma_k} X^{(k)} \to M/\Sigma_k,$$

obtained by projecting on the first factor, is a fiber bundle with fiber $X^{(k)}$. Furthermore π has a section $\sigma : M/\Sigma_k \to M \times_{\Sigma_k} X^{(k)}$ defined by $\sigma(\bar{m}) = (m, *)$ where $*$ is the basepoint of $X^{(k)}$. (This section is well defined as $*$ is fixed under the Σ_k action on $X^{(k)}$.)

Given a field $\mathbb{F}$, we will now set out to find $H^*(M \times_{\Sigma_k} X^{(k)}; \mathbb{F})$. To do that using the spectral sequence for the fiber bundle described above, we see that we first need to describe $H^*(X^{(k)}; \mathbb{F})$. So let us do that now.

Recall that if the inclusion of A into X is a cofibration, then we have a natural isomorphism $H^*(X, A; \mathbb{F}) \cong \bar{H}^*(X/A; \mathbb{F})$ where the bar signifies reduced cohomology. (See for example [Bre]).

Now if X and Y are well pointed spaces, then the inclusion of $X \vee Y$ into $X \times Y$ is a cofibration?

Thus $H^*(X \times Y, X \vee Y; \mathbb{F}) \cong \bar{H}^*(X \wedge Y; \mathbb{F})$. On the other hand, it is easy to argue that the long exact sequence in reduced cohomology for the pair $(X \times Y, X \vee Y)$ degenerates into split short exact sequences:

$$0 \to H^*(X \times Y, X \vee Y; \mathbb{F}) \to \bar{H}^*(X \times Y; \mathbb{F}) \xrightarrow{i^*} \bar{H}^*(X \vee Y; \mathbb{F}) \to 0$$

Thus $\bar{H}^*(X \wedge Y; \mathbb{F})$ is algebra isomorphic to the kernel of i^*. We are able to get a complete description of this kernel via the K*ü*nneth Theorem in the case when one of the spaces has finite dimensional $\mathbb{F}$-cohomology in each dimension. (We need this condition, to apply the cohomology version of Künneth's theorem - see [Bre]).

Using this, one easily describes the cohomology algebra of the smash product in terms of the cohomology algebras of the original spaces. We state the result as a lemma and leave the completion of the details of its proof to the reader.

Lemma 9.10. *If X, Y are path connected, well pointed spaces and at least one of them has finite dimensional $\mathbb{F}$-cohomology in each dimension, then $\bar{H}^*(X \wedge Y; \mathbb{F}) \cong \bar{H}^*(X; \mathbb{F}) \otimes \bar{H}^*(Y; \mathbb{F})$ as $\mathbb{F}$-algebras (without identity element).*

It follows from lemma 9.10 that for a path connected, well pointed space X, one has $\bar{H}^*(X^{(k)}; \mathbb{F})$ is isomorphic to the tensor product of k copies of the algebra $\bar{H}^*(X; \mathbb{F})$ with itself.

Definition 9.11. *Given a graded $\mathbb{F}$-algebra $\mathfrak{A}$ with $\mathfrak{A}^0 = 0$ (so $\mathfrak{A}$ does not have an identity element), we define $T_k(\mathfrak{A})$ to be the $\mathbb{F}$-algebra with identity obtained in the following way. We first take the tensor product of k-copies of $\mathfrak{A}$. This has nothing in degree zero. Finally, we include a 1-dimensional vector space in degree zero generated by an identity element.*

Thus we can restate our results as $H^*(X^{(k)}; \mathbb{F}) \cong T_k(\bar{H}^*(X; \mathbb{F}))$.

Fix a label space X. Now as we mentioned before, there is a natural left Σ_k action on $X^{(k)}$ induced from the left Σ_k action on X^k given by $\sigma \cdot (x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.

Since cohomology is a contravariant functor, it is easy to check that we get a right Σ_k action on $H^*(X^k; \mathbb{F})$. Let us describe this action using the K*ü*nneth theorem to identify $H^*(X^k; \mathbb{F})$ as the tensor product of k copies of $H^*(X; \mathbb{F})$. It is easy to check that an element of the form $1 \otimes \ldots \alpha \cdots \otimes 1$ where α is in the ith spot gets taken under $\sigma \in \Sigma_k$ to a similar element where α is in the $\sigma^{-1}(i)$th spot. (To check this note that such elements correspond nicely to elements in the cohomology of the k-fold wedge product of X where the statement is clear.)

This describes the right Σ_k action on $H^*(X^k; \mathbb{F})$ completely as elements of the form $1 \otimes \ldots \alpha \cdots \otimes 1$ generate this cohomology as an algebra and Σ_k acts via algebra maps.

It is convenient, to switch this right Σ_k action to a left Σ_k action via a standard procedure. Given a right action of a group G on some object X, one obtains a left action of G on X by defining

$$g \cdot x \equiv x \cdot g^{-1}.$$

Doing this the left action of Σ_k on $H^*(X^k;\mathbb{F})$ sends $1\otimes\dots\alpha\cdots\otimes 1$ where α is in the ith spot to the corresponding element where α is in the $\sigma(i)$th spot under $\sigma\in\Sigma_k$.

Thus we have in general that

$$\sigma\cdot(\alpha_1\otimes\cdots\otimes\alpha_k) = \pm\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(k)}.$$

The $\pm$ sign occurs because of the grading as the following example illustrates:

Let $k=2$ and $\sigma=(1,2)$, then

$$\begin{aligned}\sigma\cdot(\alpha\otimes\beta) &= \sigma\cdot(\alpha\otimes 1\cup 1\otimes\beta)\\ &= (\sigma\cdot(\alpha\otimes 1))\cup(\sigma\cdot(1\otimes\beta))\\ &= (1\otimes\alpha)\cup(\beta\otimes 1)\\ &= (-1)^{|\alpha||\beta|}(\beta\otimes 1)\cup(1\otimes\alpha)\\ &= (-1)^{|\alpha||\beta|}(\beta\otimes\alpha)\end{aligned}$$

Of course, this description of the left Σ_k action on $H^*(X^k;\mathbb{F})$ restricts to a description of the left Σ_k action on $H^*(X^{(k)};\mathbb{F})\cong T_k(\bar{H}^*(X;\mathbb{F}))$.

One of the main cases we will be looking at is when X is S^d, the d-sphere. In this case, we can make the sign in the left Σ_k action on $H^*(X^{(k)};\mathbb{F})$ explicit.

Definition 9.12. *A representation of Σ_k on a $\mathbb{F}$-vector space is said to be trivial if every element acts as the identity map of the vector space.*

The sign representation of Σ_k is a one dimensional $\mathbb{F}$-vector space where the even elements of Σ_k act as multiplication by 1 while the odd elements of Σ_k act as multiplication by -1.

Notice that if the characteristic of $\mathbb{F}$ is two, then the sign representation is actually trivial.

First note, that the reduced cohomology of S^d is zero except in degree d where it is one dimensional generated by α say.

Thus $\bar{H}^*((S^d)^{(k)};\mathbb{F})$ is zero except in degree kd where it is one dimensional generated by $T=\alpha\otimes\cdots\otimes\alpha$. (In fact in this case we know $(S^d)^{(k)}=S^{dk}$ but we will not use that.)

Each element in Σ_k takes T to $\pm T$. It is easy to see that if d is even, $\sigma(T)=T$ for all $\sigma\in\Sigma_k$ while if d is odd, $\sigma(T)=(-1)^{\epsilon(\sigma)}T$ where ϵ is the sign representation of Σ_k into $\{-1,+1\}\subset\mathbb{F}^*$. ($\mathbb{F}^*$ stands for the group of nonzero elements of $\mathbb{F}$.)

Thus we have shown the following useful proposition:

Proposition 9.13. *As a left Σ_k-module, $\bar{H}^*((S^d)^{(k)};\mathbb{F})$ is concentrated in degree kd and in that degree it is*
(a) A trivial one dimensional Σ_k-module if d is even.
(b) The one dimensional sign representation if d is odd.

Now let us calculate the cohomology of $M\times_{\Sigma_k}X^{(k)}$ where $X=S^d$. First recall that we had a fiber bundle $\pi: M\times_{\Sigma_k}X^{(k)}\to M/\Sigma_k$ with fiber $X^{(k)}$. Thus we have a Serre spectral sequence with

$$E_2^{p,q}=H^p(M/\Sigma_k;H^q(X^{(k)};\mathbb{F}))$$

abutting to $H^{p+q}(M\times_{\Sigma_k}X^{(k)};\mathbb{F})$. The reader is warned that the coefficients in the E_2-term are twisted. In fact from the cover $M\to M/\Sigma_k$ we get a short exact sequence of groups

$$\pi_1(M) \to \pi_1(M/\Sigma_k) \xrightarrow{\lambda} \Sigma_k.$$

The action of $\pi_1(M/\Sigma_k)$ on the cohomology of the fiber is easily checked to be given by the composition of λ and the Σ_k-action on $H^*(X^{(k)};\mathbb{F})$ described in the previous paragraphs.

Recall that the fiber bundle π had a section σ. This means that in this spectral sequence, no differentials will hit the horizontal line $q = 0$. Thus for general X, $E_2^{p,0} = E_\infty^{p,0}$ in this spectral sequence.

Now for $X = S^d$, we have E_2 is concentrated on the horizontal lines $q = 0$ and $q = kd$. Since we know no differential can hit the line $q = 0$, we conclude all diffenrentials are zero in this spectral sequence and $E_2^{*,*} = E_\infty^{*,*}$. Thus we conclude for $X = S^d$, we have an isomorphism of vector spaces

$$H^*(M \times_{\Sigma_k} (S^d)^{(k)};\mathbb{F}) \cong H^*(M/\Sigma_k;\mathbb{F}) \oplus H^{*-kd}(M/\Sigma_k; H^{kd}((S^d)^{(k)};\mathbb{F}))$$

where again recall that the last summand has a twisted coefficient.

The first summand above, is the part of $H^*(M \times_{\Sigma_k} X^{(k)};\mathbb{F})$ coming from the image of π^* or in other words coming from the image of our section. It can be shown that the section above is a cofibration, so if we form $M \rtimes_{\Sigma_k} X^{(k)}$ the space obtained from $M \times_{\Sigma_k} X^{(k)}$ by collapsing the image of the section to a basepoint, we conclude that

$$\bar{H}^*(M \rtimes_{\Sigma_k} (S^d)^{(k)};\mathbb{F}) \cong H^{*-kd}(M/\Sigma_k; H^{kd}((S^d)^{(k)};\mathbb{F}))$$

as algebras. (Here we notice that in the E_∞ term above, once we collapse the image of the section to a basepoint, we lose the row $q = 0$ and hence there are no lifting problems anymore since everything is concentrated in the row $q = kd$.)

This shows that the algebra structure of $\bar{H}^*(M \rtimes_{\Sigma_k} (S^d)^{(k)};\mathbb{F})$ is trivial, i.e., the product of any two elements is zero.

10. Lie Algebras and Kohno-Falk-Randell Theory

The purpose of this section is to consider the functor from groups to Lie algebras given by sending a group G to the Lie algebra obtained from the descending central series for G.

The Lie algebras associated in this way to some pure braid groups as well as the fundamental groups of orbit configuration spaces appear in several different mathematical contexts.

In this section, a very useful tool for the analysis of these Lie algebras, obtained by T. Kohno [**?**], and Falk, and Randell [**?**] is described. They used this tool to analyze the beautiful case where G is the kth pure Artin braid group. This general theory is described below along with several examples related to braid groups and elliptic curves.

We begin by defining some preliminary group theoretic concepts.

Definition 10.1. *Given two subgroups H, K of a group G, we define $[H,K]$ to be the subgroup of G generated by commutators $[h,k] = h^{-1}k^{-1}hk$ where h ranges over the elements of H and k ranges over the elements of K.*

It is easy to check that $[H,K]$ is normal (characteristic) in G if H and K are normal (characteristic) in G. (Recall a subgroup of G is called characteristic if it is invariant under every automorphism of G.)

Definition 10.2. *We define the descending central series of G,*

$$G = \Gamma^1(G) \geq \Gamma^2(G) \geq .... \geq \Gamma^n(G) \geq \dots$$

inductively by $G = \Gamma^1(G)$ and $\Gamma^n(G) = [G, \Gamma^{n-1}(G)]$ for $n > 1$.

The following proposition collects some elementary facts about this series, the proofs of which are easy and left to the reader: [references: see Magnus, Karass, Solitar, or Michael Vaughn-Lee]

Proposition 10.3. *Let $\Gamma^n(G)$ denote the nth term in the descending central series, then:*
(1) The group $\Gamma^n(G)$ is a normal (in fact characteristic) subgroup of G.
(2) The quotient group $E_0^n(G) = \Gamma^n(G)/\Gamma^{n+1}(G)$ is abelian for all $n \geq 1$.
(3) The map of sets given by the commutator $<,>: G \times G \to G$ by defining

$$< x, y >= x^{-1}y^{-1}xy$$

induces a well-defined bilinear pairing

$$[\cdot,\cdot] : E_0^n(G) \otimes_{\mathbb{Z}} E_0^m(G) \to E_0^{n+m}(G)$$

which is alternating, i.e.,

$$[a, a] = 0$$

for all a, and satisfies the Jacobi identity, i.e.,

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

for all a, b, c.
(4) Thus $E_0^*(G)$ is a quasi-graded Lie algebra in the sense that if a has degree n and b has degree m, then $[a, b]$ has degree $n+m$. The reader is warned that this notion of quasi-graded Lie algebra is different from the notion of a graded Lie algebra typically used in topology which is defined below for completeness.

Definition 10.4. *A graded Lie algebra is a graded abelian group $E^* = \oplus_{i\in\mathbb{Z}}E^i$ together with a bilinear bracket $[\cdot,\cdot] : E^* \otimes E^* \to E^*$ satisfying:*

$$[a, b] = (-1)^{|a||b|}[b, a]$$

and

$$(-1)^{|a||c|}[[a, b], c] + (-1)^{|b||a|}[[b, c], a] + (-1)^{|c||b|}[[c, a], b] = 0$$

for all homogeneous $a, b, c \in E^*$. (Here $|a|$ denotes the degree of a etc.)

The motivation for the definition of a graded Lie algebra given above is that the homotopy groups $\{\pi_n(X)|n \geq 2\}$ of a well-pointed space X fit together to give a graded Lie algebra under the Whitehead product. (See [Bre].)

Consider the Lie algebra $E_0^*(G)$ obtained from the descending central series for the group G as described in proposition 10.3. For each positive integer q, there is a canonical graded Lie algebra $E_0^*(G)_q$ obtained from $E_0^*(G)$, which will be useful in the next section, and which is defined as follows.

Definition 10.5. *Fix a positive integer q and let $\Gamma^n(G)$ denote the nth stage of the descending central series for G. Define*

$$E_0^i(G)_q = \begin{cases} E_0^n(G) \text{ if } i = 2nq \\ 0 \text{ if } i \neq 0 \text{ mod } 2q \end{cases}$$

Finally define the Lie bracket on $E_0^*(G)_q$ to be that induced from the one of $E_0^*(G)$.

We are now ready to look at the main theorem of this section.

Theorem 10.6 (T. Kohno, Falk-Randell)**.** *Let*

$$1 \to A \to B \to C \to 1$$

be a split short exact sequence of groups such that the conjugation action of C on $H_1(A)$ is trivial.

Then there is a short exact sequence of Lie algebras

$$0 \to E_0^*(A) \to E_0^*(B) \to E_0^*(C) \to 0$$

which is split as a sequence of abelian groups. (Thus there is an isomorphism of abelian groups $E_0^n(A) \oplus E_0^n(C) \cong E_0^n(B)$ but this isomorphism need not preserve the Lie algebra structure.)

Proof. This proof follows that of Falk-Randell [**?**] and Xicoténtcatl [**?**]. Consider the extension

$$1 \to A \xrightarrow{j} B \xrightarrow{p} C \to 1$$

and let $\sigma : C \to B$ be a splitting for p. Observe that $p(b\sigma(p(b^{-1}))) = 1$ for all $b \in B$, so there exists a unique element $a \in A$ with $j(a) = b(\sigma(p(b^{-1})))$ for all $b \in B$. Thus, there is a well-defined function (which is not necessarily a homomorphism) $\tau : B \to A$ defined by the formula $\tau(b) = j^{-1}(b\sigma(p(b^{-1})))$.

Notice that the trivial action of C on $H_1(A)$ gives

(1) $cac^{-1} = ax$ for a in A, and x in $[A, A]$,
(2) $[B, A]$ is a subgroup of $[A, A]$, and
(3) $[\Gamma^n(B), \Gamma^m(A)]$ is a subgroup of $\Gamma^{m+n}(A)$, and
(4) $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$.

Since $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$, for all n, there is a well-defined induced map of sets $\tau : E_0^n(B) \to E_0^n(A)$ where τ is defined on an equivalence class of b by the formula $\tau([b]) = \tau(b)$ (as $\tau(bv) = bv.\sigma p((bv)^{-1})= \tau(b).\Gamma$ where v and Γ are in $\Gamma^{n+1}(B)$.

Furthermore if b is in $ker(p) \cap \Gamma^n(B)$, then $\tau(b) = b$. Thus τ restricts to a function $\tau|_{ker(p)\cap\Gamma^n(B)} : ker(p) \cap \Gamma^n(B) \to \Gamma^n(A)$, and the homomorphism $j : \Gamma^n(A) \to ker(p)\cap = \Gamma^n(B)$ is a group isomorphism. Thus, there is an exact sequence of groups $1 \to \Gamma^n(A) \to \Gamma^n(B) \to \Gamma^n(C) \to 1$ which is split by the existence of σ.

Furthermore, if $[b]$ is in the kernel of $E_0^n(p) : E_0^n(B) \to E_0^n(C)$, then $E_0^n(\tau[b]) = [b]$. Hence, there is a split short exact sequence $0 \to E_0^n(A) \to E_0^n(B) \to E_0^n(C) \to 0$. The theorem follows from the above.

□

The additive decomposition in theorem 10.6 may not necessarily preserve the underlying Lie algebra structure. The Lie product is sometimes ”twisted”, and quite interesting, as shall be seen in examples below. We will now look at some examples which demonstrate that both hypotheses on the theorem are required (The existance of the splitting on the sequence of groups and the trivial action on homology.)

Example 1: Let $F[S]$ denote the free group on a set S. Then $E_0^*(F[S])$ is isomorphic to the free Lie algebra on S, denoted by $L[A_S]$. This is defined by letting A_S be the free abelian group with basis S, and then defining $L[A_S]$ to be the smallest sub-Lie algebra of the tensor algebra $T[A_S]$ containing A_S. [P. Hall, W. Magnus, J. P. Serre].

Example 2: Let G denote the kth pure Artin braid group P_k. Fix a free abelian group V_k with basis given by elements $B_{i,j}$ for $1 \leq i < j \leq k$. Let $\mathcal{L}_k$ denote the quotient of the free Lie algebra $L[V_k]$ generated by V_k, modulo the following "infinitesimal braid relations":

(1) $[B_{i,j}, B_{s,t}] = 0$ if $\{i,j\} \cap \{s,t\} = \phi$,
(2) $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ for $1 \leq j < t < i \leq k$, and
(3) $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ for $1 \leq j < t < i \leq k$.

Then $E_0^*(P_k)$ is isomorphic to $\mathcal{L}_k$. [T. Kohno, Falk-Randell].

Example 3: Consider the orbit configuration space $F_G(M,k)$ where $M = \mathbb{C}$, the complex numbers. Let G be the standard integral lattice $L = \mathbb{Z} + i\mathbb{Z}$, acting by translation on $\mathbb{C}$. Then $F_L(\mathbb{C},k)$ is a $K(\pi,1)$ which is studied in [Cohen,Xicoténcatl]. Let $F_L(\mathbb{C},k)$ be defined as above. Picking a parametrized lattice $\mathbb{Z} + \omega\mathbb{Z}$ gives an analogous orbit configuration space associated to an elliptic curve. These will be addressed elsewhere.

(1) The symmetric group Σ_k acts on $F_L(\mathbb{C},k)$ and the orbit space $F_L(\mathbb{C},k)/\Sigma_k$ is homeomorphic to the subspace of monic polynomials of degree k, $p(z) \in \mathbb{C}[z]$, with the property that the difference of any two roots of $p(z)$, α_i, α_j, lies outside of the Gaussian integers.
(2) It is the complement on $\mathbb{C}^k$ of the infinite (affine) hyperplane arrangement
$$\mathcal{A} = \{H_{i,j}^{\sigma} \mid 1 \leq j < i \leq k,\ \sigma \in L\}$$
where $H_{i,j}^{\sigma} = \ker(z_i - z_j - \sigma)$ and σ ranges over the lattice L.
(3) It is an L-cover of the ordinary configuration space of k points in the torus $T = S^1 \times S^1$. This is a special case of the results in [Xicoténcatl, thesis] where it is proven that there exists a principal bundle
$$L^k \to\ F_L(\mathbb{C},k) \to\ F(T,k).$$
(4) The space $F_L(C,k)$ is a $K(\pi,1)$-space.
(5) The fibration $F_L(C,k) \to\ F_L(C,k-1)$ has
(i) trivial local coefficients in homology, and
(ii) a cross-section.
(6) Thus by theorem 10.6, the Lie algebra attached to the descending central series of $\pi_1(F_L(\mathbb{C},k))$ is additively isomorphic to the direct sum $\bigoplus_{1<i\leq k} L[i]$ where $L[i]$ is the free Lie algebra generated by elements $B_{i,j}^{\sigma}$ for fixed i with $1 \leq j < i \leq k$, and σ runs over the elements of the lattice L.
(7) The relations are
$[B_{i,j}^{\sigma}, B_{k,i}^{\tau}] = [B_{k,i}^{\tau}, B_{k,j}^{\tau+\sigma}]$
$[B_{i,j}^{\sigma}, B_{k,j}^{\tau}] = [B_{k,j}^{\tau}, B_{k,i}^{\tau-\sigma}]$

The next list of examples gives short exact sequences of groups such that the conclusion of the Kohno-Falk-Randell theorem fails, and where one of the hypotheses in the theorem does not hold.

Example 4: Consider a split short exact sequence of groups

$$1 \to A \to B \to C \to 1$$

where both A and C are abelian.

Thus if $n \geq 2$, both $E_0^n(A)$, and $E_0^n(C)$ are trivial. However, it may well be the case that B is not abelian, for example if B is a nontrivial semi-direct product of A and C, then $E_0^2(B)$ will be non-trivial and hence is not the sum of $E_0^2(A)$ with $E_0^2(C)$, thus spoiling the conclusion of the Kohno-Falk-Randell theorem. Notice here that in this case C will act nontrivially on $H_1(A) = A$.

The simplest example of this sort is given by taking the symmetric group on 3 letters Σ_3 as B. If we take A to be the normal Sylow-3 group of order 3 and C to be the group of order 2, the group extension formed by A, B and C is split, but the action of C on the first homology group of A is non-trivial. In fact $E_0^2(\Sigma_3) = \mathbb{Z}/3\mathbb{Z}$.

Example 5: Next consider group extensions
$1 \to A \to B \to C \to 1$ where $A = \mathbb{Z}/2\mathbb{Z}$. Since the only automorphism of $\mathbb{Z}/2\mathbb{Z}$ is the identity, the action of C on $H_*(A)$ is always trivial. If this extension fails to split, then the conclusion of the Kohno-Falk-Randell theorem may be spoiled. For example we may take $A = C = \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/4\mathbb{Z}$. Then $E_0^1(B) \neq E_0^1(A) \oplus E_0^1(C)$.

Another less trivial example is provided by non-abelian extraspecial 2-groups where $C = (\mathbb{Z}/2\mathbb{Z})^n$. In this case $E_0^2(B)$ is again isomorphic to $\mathbb{Z}/2\mathbb{Z}$, but $E_0^2(A) \oplus E_0^2(C)$ is the trivial group. Two more specific examples where B is non-abelian, and $C = (\mathbb{Z}/2\mathbb{Z})^2$ are given by D_8, the dihedral group of order 8, and Q_8, the quaternion group of order 8.

Example 6: Assume that the group C in the extension $1 \to A \to B \to C \to 1$ is free. Then this extension is split. This setting gives infinite examples where the conclusion of the Kohno-Falk-Randell theorem may be spoiled. A specific example is given by C the free group generated by a finite set S of cardinality n, B is the free group generated by the coproduct of the two non-empty sets $S \amalg T$ where T has cardinality 1 with B $= F[S \amalg T]$ and where the map $p : B \to C$ given by the natural projection.

Then the above extension is split, but $E_0^1(A)$ is a countably infinitely generated free abelian group, while $E_0^1(B)$ is a free abelian group of rank n+1. Thus the natural map $E_0^1(A) \to E_0^1(B)$ has a kernel.

Example 7: If M is a punctured surface of genus greater than 0, and $k \geq 2$ then the fibrations $F(M, k) \to F(M, k-1)$ have sections, but the local coefficient system is non-trivial (as can be seen by inpsection of the relevant Dehn twist). Thus the Lie algebras attached to the descending central series for the pure braid groups of these surfaces is not clear. One case above addresses this structure by considering the group that is the kernel of the map
$\pi_1 F(M, k) \to \pi_1 M^k$ that is induce by the inclusion map.

When M is a torus, the kernel of this last map satisfies the hypotheses for the Kohno-Falk-Randell theorem. The resulting Lie algebra is given above [CX].

When M is any closed surface of genus at least one, it seems likely that this kernel always satisfies the hypotheses of the Falk-Randell-Kohno theorem.

Let H denote the upper 1/2-plane, and let Γ be a subgroup of $SL(2, Z)$ that acts properly discontinuously on H. One might conjecture that the fundamental groups of the orbit configuration spaces $\pi_1 F_\Gamma(H, k)$ satisfy the hypotheses of the Kohno-Falk-Randell theorem, and thus the conclusion.

By [**?**], there is a short exact sequence of groups

$$1 \to \pi_1 F_\Gamma(H, k) \to \pi_1 F(M, k) \to \pi_1(M^k) \to 1$$

where $H/\Gamma = M$. The point of this is that the orbit configuration space has a "nice" associated Lie algebra. The above extension tweezes apart two different phenomona in these Lie algebras.

11. Loop spaces of configuration spaces, and Lie algebras

This section is about 2 possibly different constructions which are in fact the same. The subject of this section is loop spaces of configuration spaces and their relationship to the Lie algebra attached to the descending central series of the pure braid group as described in the previous section. The purpose of this section is to show that these Lie algebras, apart from a formal degree shift, are given by the homotopy groups of configuration spaces for points in complex n-space, $n > 1$, modulo torsion. This theorem, first proven in [**?**], and subsequently in [CG] is a special case of a more general result , and which applies to further analogues of pure braid groups. One example, the space of monic polynomials where the differences of the roots lie ouside of the Gaussian integers provides another example, as well as certain choices of orbit configuration spaces are also described below.

Some additional discussion (no proofs !) are given concerning other related constructions which exhibit properties like braid groups, and appear in the space of loops on a configruation space. These loop spaces may be thought of as braids on a manifold, or as trajectories of distinct particles moving through a time parameter that start and quit in the same position.

These constructions also "fit" into several dfferent contexts. One of which is that these spaces admit interpretations in terms of Vassiliev invariants of braids, and knots. This subject will not be addressed here; some information is given in [CG]. Indeed, one motivation for including this information here is that loop spaces of configuration spaces keep track of paths of distinct particles parametrized by time as they move through a manifold, and are essentially braids on a manifold. There is a rich homological structure attached to these paths, as well as a close connection to invariants of knots.

Recall the regrading of Lie algebras $E_0^*(G)_m$ for $m > 0$ given in the previous section where $E_0^*(G)$ denotes the Lie algebra attached to the descending central series of a discrete group G.

Theorem 11.1. *If $m \geq 1$, then the homology of the loop space of the configuration space $\Omega F(R^{2m+2}, k)$ is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(P_k)_m$. Furthermore,*

(1) *the image of the classical Hurewicz homomorphism $\pi_*(\Omega F(R^{2m+2}, k)) \to H_*(\Omega F(R^{2m+2}, k))$ is isomorphic to $E_0^*(P_k)_m$,*
(2) *the Hurewicz homomorphism induces an isomorphism of graded Lie algebras*
$\pi_*(\Omega F(R^{2m+2}, k))/torsion \to \ PrimH_*(\Omega F(R^{2m+2}, k))$
where Prim(.) denotes the module of primitive elements, and the Lie algebra structure of the source is given by the classical Samelson product.

Namely, the homotopy groups of the loop space of the configuration space of k points in an even dimensional euclidean space R^{2m+2}, modulo torsion, admits the structure of a graded Lie algebra induced by the classical Samelson product. That Lie algebra is isomorphic to $E_0^*(P_k)_m$ as described in the previous section.

Similar results apply to other analogues of pure braid groups. Further work of [X], [DC], and the first author show that an analogous theorem holds for some other groups that are "close" to braid groups arising from some fibred $K(\pi, 1)$ hyperplane arrangements. Analogous results hold for "orbit configuration spaces" for groups acting freely on the upper 1/2-plane, and for some lattices acting on C [CX].

Here, consider $F_G(M, k)$ in the case when $M = C$, the complex numbers, and G is the integral lattice $L = Z + iZ$, acting by translation on C. One of the consequences of the theorem below is that the Lie algebra obtained from the fundamental group of the associated orbit configuration space also gives the Lie algebra obtained form the higher homotopy groups of the "higher dimensional analogues" of this arrangement. Two examples illustrating this behavior are [X], [DC], and are described here as well as general theorem about analogous spaces.

Theorem 11.2. *Let $F_L(C, k)$ be defined as above.*

(1) *The symmetric group Σ_k acts on $F_L(C, k)$ and the orbit space $F_L(C, k)/\Sigma_k$ is homeomorphic to the subspace of monic polynomials of degree k, $p(z) \in C[z]$, with the property that the difference of any two roots of $p(z)$, α_i, α_j, lies outside of the Gaussian integers.*
(2) *It is the complement in C^k of the infinite (affine) hyperplane arrangement*
$$\mathcal{A} = \{H^{\sigma}_{i,j} \mid 1 \leq j < i \leq k,\ \sigma \in L\}$$
where $H^{\sigma}_{i,j} = \ker(z_i - z_j - \sigma)$.
(3) *It is an L^k-cover of the ordinary configuration space of k points in the torus $T = S^1 \times S^1$. This is a special case of the results in* [**?**]*, which gives the existence of a principal bundle*
$$L^k \longrightarrow F_L(C, k) \longrightarrow F(T, k).$$
(4) *The space $F_L(C, k)$ is a $K(\pi, 1)$.*
(5) *The fibration $F_L(C, k) \to \ F_L(C, k-1)$ has (i) trivial local coefficients in homology, and (ii) a cross-section.*
(6) *Thus the Lie algebra given by the associated graded for the descending central series of $\pi_1(F_L(C, k))$ is additively isomorphic to the direct sum*

$\bigoplus_{1<i\leq k} L[i]$ where $L[i]$ is the free Lie algebra generated by elements $B^{\sigma}_{i,j}$ for fixed i with $1 \leq j < i \leq k$, and σ runs over the elements of the lattice L.

(7) *The relations are*
$[B^{\sigma}_{i,j}, B^{\tau}_{k,i}] = [B^{\tau}_{k,i}, B^{\tau+\sigma}_{k,j}]$
$[B^{\sigma}_{i,j}, B^{\tau}_{k,j}] = [B^{\tau}_{k,j}, B^{\tau-\sigma}_{k,i}]$

(8) *The integral homology of $F_L(C,k)$ is additively given by*

$$H_*F_L(C,\ell) \cong H_*(C_1) \otimes H_*(C_2) \otimes \cdots \otimes H_*(C_{k-1})$$

where C_i is the infinite bouquet of circles $\bigvee_{|Q^L_i|} S^1$ and Q^L_i as defined in the beginning of next section.

Consider the "orbit configuration space" $F_L(C \times R^n, k)$ where L operates diagonally on $C \times R^n$, and trivially on R^n.

Theorem 11.3. *Assume that $q \geq 1$.*

(1) *The loop space $\Omega F_L(C \times R^{2q}, k)$ is homotopy equivalent to the product*

$$\prod_{1\leq i\leq k-1} \Omega(C \times R^{2q} - Q^L_i)$$

(although this product decomposition is not multiplicative).

(2) *The integral homology of $\Omega F_L(C \times R^{2q}, k)$ is isomorphic to*

$$\bigotimes_{1\leq i\leq k-1} H_*(\Omega(C \times R^{2q} - Q^L_i))$$

as a coalgebra.

(3) *The Lie algebra of primitives is isomorphic to the Lie algebra given by $\pi_*(\Omega F_L(C \times R^{2q}, k))/torsion$.*

(4) *The Lie algebra of of primitive elements in the homology of $\Omega F_L(C\times R^{2q}, k)$ is a direct sum of free (graded) Lie algebras $\bigoplus_{1<i\leq k} L[i]$ where $L[i]$ is the free graded Lie algebra generated by elements $B^{\sigma}_{i,j}$ of degree $2q$ for fixed i with $1 \leq j < i \leq k$, and σ runs over the elements of the lattice L. The relations are*
$[B^{\sigma}_{i,j}, B^{\tau}_{k,i}] = [B^{\tau}_{k,i}, B^{\tau+\sigma}_{k,j}]$
$[B^{\sigma}_{i,j}, B^{\tau}_{k,j}] = [B^{\tau}_{k,j}, B^{\tau-\sigma}_{k,i}]$

(5) *The Lie algebras $\pi_*(\Omega F_L(C^q, k))$ modulo torsion, and $E^*_0(F_L(C,k))_q$ are isomorphic as Lie algebras.*

Some of these Lie algebras occur for comparatively general reasons as exemplified by the next theorem. The following general theorem does not specifiy the structure constants for the underlying Lie algebra, but shows that the Lie algebras addressed above fit in a wider context.

Theorem 11.4. *Assume that $n \geq 3$. Let $X(R^n, k) \to X(R^n, k-1)$ be a fibration which satisfies the following properties:*

(1) *The fibre of* $X(R^n,k) \to X(R^n,k-1)$ *is* $R^n - S_k$ *where* S_k *is a discrete subspace of* R^n *of fixed (not neccessarily finite) cardinality depending on* k.
(2) *Each fibration* $X(R^n,k) \to X(R^n,k-1)$ *admits a cross-section.*
(3) *The space* $X(R^n,1)$ *is* R^n *with* $n \geq 3$.

Then

(1) *There is a homotopy equivalence* $\Omega X(R^n,k) \to \prod_{1\leq i\leq k-1} \Omega(R^n - S_i)$.
(2) *The homology of* $\Omega X(R^n,k)$ *is torsion free, and is isomorphic to* $\bigotimes_{1\leq i\leq k-1} H_*(\Omega(R^n - S_i))$ *as a coalgebra.*
(3) *The module of primitives in the integer homology of* $\Omega X(R^n,k)$ *is isomorphic to*
$E_0^*(\pi_*(\Omega X(R^n,k)))$ *modulo torsion as a Lie algebra.*

In many examples which arise from fibre type arrangements, a similar conclusion holds as that given in Theorem 1.1, and Theorem 1.3 part (5) above. Namely, there are $K(G,1)'s$ in "good" cases where $\pi_*(\Omega X(R^n,k))$ modulo torsion is isomorphic to $E_0^*(G)_q$. The last section of this article contains some speculation as to where, and how these structures fit. It is a general theorem that if X is a 1-connected CW complex, there is a functor $\Theta(X) = K(G_X,1)$ where G_X is a filtered group such that the associated graded Lie algebra tensored with the rational numbers gives the so-called "homotopy Lie algebra" for the loop space of the rationalization of X. [CS]. This theorem admits some overlap with work of T. Kohno, and T. Oda [KO] on the descending central series of the pure braid group of an algebraic curve.

12. Proof of the theorem 1.4

Recall that a multiplicative fibration with section is homotopy equivalent to a product. Thus $\Omega X(R^n,k)$ is homotopy equivalent to $\Omega X(R^n,k-1)\times\Omega(R^n - S_{k-1})$, and the first part of the theorem follows by induction.

The second part of the theorem follows from the *Künneth* theorem, and part 1 of the theorem.

Since $R^n - S_{k-1}$ has the homotopy type of a (possibly infinite) bouquet of $(n-1)$-spheres, the homology of its loop space follows from the Bott-Samelson theorem. In this case, it is well-known that there are isomorphisms of Lie algebras $\pi_*(\Omega(R^n - S_{k-1})/torsion \to PrimH_*(\Omega(R^n - S_{k-1})$.

Furthermore, the existence of sections implies that the Hurewicz homomorphism $\pi_*(\Omega X(R^n,k) \to PrimH_*(\Omega X(R^n,k))$ is a surjection. Since this map is an injection, the theorem follows.

13. Proof of Theorem 1.3

Consider the fibration with section $F_L(C\times R^{2q},k) \to F_L(C\times R^{2q},k-1)$. The fibre of this map is $C\times R^{2q} - Q^L_{k-1}$. By Theorem 1, the conclusions of Theorem

3 all follow except possibly the last two which state the precise extension of Lie algebras.

To finish, it suffices to prove parts (4), and (5) of the theorem by a direct comparison of the two Lie algebras $\pi_*(\Omega F_L(C \times R^{2q}, k))/torsion$, and $E_0^*(F_L(C,k))_q$.

Thus define maps analogous to those in the proof of the relations for the Lie algebra attached to the descending central series above: $F_i : S^{2q+1} \times S^{2q+1} \to F_L(C \times R^{2q}, 3)$ by the formula

$$F_1(z,w) = (q_1, q_1 + \sigma + \frac{z}{8}, q_1 + \tau + \frac{w}{16})$$
$$F_2(z,w) = (q_1, q_1 + \sigma + \frac{z}{8}, q_1 + \sigma + \tau + \frac{z}{8} + \frac{w}{16}).$$

Consider the loopings of these maps $\Omega(F_i) : \Omega(S^{2q+1} \times S^{2q+1}) \to \Omega(F_L(C \times R^{2q}, 3))$. Notice that the fundamental cycles in degree $2q$ for the integer homology of $\Omega(S^{2q+1} \times S^{2q+1})$ commute. Thus it suffices to calculate the image of the fundamental cycles in the homology of $\Omega(F_L(C \times R^{2q}, 3))$. This gives the precise relations as stated in parts (4)-(5) of Theorem 3 which follows at once.

Consider the Lie algebra obtained from the descending central series for the group G. For each strictly positive integer q, there is a canonical (and trivially defined) graded Lie algebra $E_0^*(G)_q$ attached to the one obtained from the descending central series for G, and which is defined as follows.

(1) Fix a strictly positive integer q.
(2) Let $\Gamma^n(G)$ denote the n-th stage of the descending central series for G.
(3) $E_0^{2nq}(G)_q = \Gamma^n(G)/\Gamma^{n+1}(G)$,
(4) $E_0^i(G)_q = \{0\}$, if i is non-zero modulo $2q$, and
(5) the Lie bracket is induced by that for the associated graded for the $\Gamma^n(G)$.

The main theorem here is an interpretation of the results in [CG], and [FH] concerning the homology of the loop space of configuration spaces.

Theorem 13.1. *If $m \geq 1$, then the homology of the loop space of the configuration space $\Omega F(R^{2m+2}, k)$ is isomorphic to the universal enveloping algebra of the graded Lie algebra $E_0^*(P_k)_{2m}$. Furthermore, the following are satisfied.*

(1) *The image of the classical Hurewicz homomorphism $\pi_*(\Omega F(R^{2m+2}, k)) \to H_*(\Omega F(R^{2m+2}, k))$ is isomorphic to $E_0^*(P_k)_{2m}$,*
(2) *the Hurewicz homomorphism induces an isomorphism of graded Lie algebras where where Prim(.) denotes the module of primitive elements with the Lie algebra structure of the source induced by the classical Samelson product:* $\pi_*(\Omega F(R^{2m+2}, k))/torsion \to \ PrimH_*(\Omega F(R^{2m+2}, k))$
(3) *If $q \geq 1$, the Euler-Poincare' series for the homology of $H_*(\Omega F(R^{q+2}, n); \mathbb{Z})$ is given as follows:* $[(1-t^q)(1-2t^q)....(1-(n-1)t^q)]^{-1}$.

Namely, the homotopy groups of the loop space of the configuration space of k points in an even dimensional euclidean space R^{2m+2}, modulo torsion, admits the structure of a graded Lie algebra induced by the classical Samelson product. That Lie algebra is isomorphic to $E_0^*(P_k)_m$, the Lie algebra that is "universal" for the Yang-Baxter-Lie relations.

The theorem above appears to be related to beautiful results of T. Kohno and others [H,K,K2,FR,W] who consider the relationship between Vassiliev invariants of braids as well as other structures. Kohno has recently considered the homology of the loop space for configurations in R^3. In particular, the universal enveloping algebra of $E_0^*(P_k)$ regarded as a graded abelian group has Euler-Poincare' series given by $[(1-t)(1-2t)....(1-(n-1)t)]^{-1}$ while the Euler-Poincare' series for the homology of $H_*(\Omega F(R^{q+2},n);\mathbb{Z})$ is $[(1-t^q)(1-2t^q)....(1-(n-1)t^q)]^{-1}$.

Given a fibre type arrangement $X(k,R^n)$ with fibrations $X(k,R^n) \to X(k-1,R^n)$ having sections with fibre given by $R^n - S$ where S is a discrete set, consider the graded Lie algebra $E_0^*(\pi_1(X(k,R^2)))_q$. A conjecture stated in [CX] suggests that $E_0^*(\pi_1(X(k,R^2)))_q$ is isomorphic to the Lie algebra of primitive elements in the homology of $\Omega X(k,R^{2q+2})$ for many choices of $X(k,R^n)$.

There is more to this story. Interesting examples are given by the pure braid groups for "orbit configuration spaces" in C^*. In particular, the "orbit configuration space" $F_G(M,k)$ is the space of ordered k tuples of points in M that lie on distinct orbits of a free G action on M. Work of M. Xicoténcatl [X], and D. Cohen [C] imply this conjecture for the associated pure braid groups for "orbit configuration spaces" in C^*.

Namely, this conjecture is correct for spaces $F_{Z/qZ}(C^n-\{0\},k)$ where Z/qZ is a finite cyclic group acting freely by rotations on $C^n-\{0\}$. If $n>1$, the Lie algebra obtained from the homotopy groups modulo torsion of the loop spaces for these choices of "orbit configuration spaces" is isomorphic to the Lie algebra obtained from the descending central series for $\pi_1 F_{Z/qZ}(C^n-\{0\})$ [C],[X].

Further work of M. Xicoténcatl, D. Cohen, and the first author shows that an analogous theorem holds for some other groups that are "close" to braid groups, and arise from some fibred $K(\pi,1)$ hyperplane arrangements. Some similar results hold for "orbit configuration spaces" for groups acting freely on the upper 1/2-plane, and for some lattices acting on C [CX].

There are related groups that share some common properties here. Let $Hom^{coalg}(T[v],H)$ denote the set of coalgebra morphisms with source given by the tensor algebra over the integers with a single primitive algebra generator v in degree 1. Furthermore, the target H is a Hopf algebra with conjugation (antipode). Recall that this set is naturally a group with multiplication induced by the coproduct for the source and product for the target with inverses induced by the conjugation in H [MM].

There are groups $Hom^{coalg}(T[v], H_*\Omega F(M,k))$ where M is a manifold for which the homology of its' loop space is torsion free. When M is euclidean space, this group is not isomorphic to the pure braid group, but in this case the associated Lie algebra resembles $E_0^*(P_k)_q$. The first approximation in this direction is as follows.

Theorem 13.2. *[S]*
If H is isomorphic to a tensor algebra generated by a rational vector space of dimension q concentrated in a fixed even degree that is strictly greater than 0, S is a set of cardinality q, and $F[S]_M$ is the Malĉev completion of F[S], then there is an isomorphism of groups

$$F[S]_M \to \ Hom^{coalg}(T[v], H).$$

A similar calculation gives the following.

Theorem 13.3. *[CS] If $m \geq 1$, the group $Hom^{coalg}(T[v], H_*\Omega F(R^{2m+2}, k))$ is filtered such that the associated graded is a graded Lie algebra which when tensored with $\mathbb{Q}$ is isomorphic to $E_0^*(P_k)_{2m} \otimes \mathbb{Q}$*

The group $Hom^{coalg}(T[v], H_*\Omega(X))$ accepts homomorphisms from the group of homotopy classes of pointed maps $[\Omega S^2, \Omega(X)]$. The point is that this construction provides a group theoretic analogue of the classical Hurewicz homomorphism, and that these groups are more primitive versions of homotopy groups. The actual Hurewicz map is the induced map on the level of associated graded groups. In addition, one obtains further braid-like groups by replacing euclidean space by other manifolds M.

14. Proof of Theorem 1.1

The main theorem is essentially proven in [CG], and [**?**] except for the statement about the module of primitives. In that article, there are maps constructed
$B_{i,j} : S^{n-2} \to \ \Omega F(R^n, k)$
for $k \geq i > j \geq 1$ such that the image of the fundamental cycles in the homology of $\Omega F(R^n, k)$ are (1) non-zero, and (2) the homology of $\Omega F(R^n, k)$ is generated by these classes as a Hopf algebra. Furthermore, Samelson products of the generators map to the analogous Lie element in the homology of $\Omega F(R^n, k)$ by the above construction.

These relations arise as follows. Define maps

$$\gamma_i : S^{n-1} \times S^{n-1} \to F(\mathbb{R}^n, 3)$$

by the following formulas where $\|u\| = \|v\| = 1$.

(i): $\gamma_1(u, v) = (0, u, 2v)$, and
(ii): $\gamma_2(u, v) = (0, 2u, v)$.

Next, recall that the class $A_{i,j}$ is defined by the equation

$$A_{i,j} = \pi_{i,j}^*(\iota)$$

where $\pi_{i,j} : F(\mathbb{R}^n, k) \to F(\mathbb{R}^n, 2)$ denotes projection on the (i, j) coordinates and ι is a fixed fundamental cycle for S^{n-1} [**?**, **?**].

Lemma 14.1. *If $n \geq 2$, then*

(1) $\gamma_1^* A_{2,1} = \iota \otimes 1$,
(2) $\gamma_1^* A_{3,1} = 1 \otimes \iota$,
(3) $\gamma_1^* A_{3,2} = 1 \otimes \iota$,
(4) $\gamma_2^* A_{2,1} = \iota \otimes 1$,
(5) $\gamma_2^* A_{3,1} = 1 \otimes \iota$, *and*
(6) $\gamma_2^* A_{3,2} = \iota \otimes 1$.

Proof. Since all the cases above are similar, one will be worked out. Notice that γ_1 composed with the projection $\pi_{1,2}$ is homotopic to first coordinate projection from the product $S^{n-1} \times S^{n-1}$ to S^{n-1}. This suffices. $\square$

Direct dualization gives the following lemma with the details of proof omitted.

Lemma 14.2. *If $n \geq 2$, then*

(1) $[B_{i,j}, B_{s,t}] = 0$ *if* $\{i, j\} \cap \{s, t\} = \phi$,
(2) $[B_{i,j}, B_{i,t} + (-1)^n B_{t,j}] = 0$ *for* $1 \leq j < t < i \leq k$, *and*
(3) $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *for* $1 \leq j < t < i \leq k$.

The relations in Lemma 4.2 above are called the (graded) infinitesimal braid relations.

If n = 2m+2 with $n > 2$, the loop space of of a finite bouquet of (n-1) spheres is homotopy equivalent to a product of loop spaces of odd dimensional spheres by the Hilton-Milnor theorem. Furthermore, the module of primitives for ΩS^{2k+1} is given by a copy of the integers in degree 2k. Thus by the Hilton-Milnor theorem, the module of primitives is given by the Lie algebra generated by the $B_{i,j}$.

The Lie algebra generated by the $B_{i,j}$ is in the Hurewicz image as the Samelson product of two elements x, and y in homotopy map to the bracket $[\phi(x), \phi(y)]$ in homology where ϕ denotes the Hurewicz homomorphism. Thus the Hurewicz homomorphism surjects to the module of primitives, and restricts to a monomorphism on the torsion free summand of homotopy groups. The kernel is precisely the torsion in the homotopy as the homology is torsion free.

The theorem follows.

Remarks. There are analogous relations satisfied for the homology of the loop spaces of many other configuration spaces of ordered k tuples of points in certain manifolds M.

Define the "extended infinitesimal braid relations" as follows:

(1) $[B_{i,j}, x_s] = 0$ if $\{i, j\} \cap \{s\} = \phi$,
(2) $[B_{i,j}, x_i + x_j] = 0$.

These relations are satisfied in the homology of the loop space of the configuration space based on the manifold M given by the product $R^1 \times N$. In this case, the loop space splits as product where one factor is $(\Omega N)^k$, and the classes x_i above arise from a class in the $i - th$ factor of ΩN. Details are given in [CG].

References

[Bre] Bredon, *Topology and Geometry*, Springer Verlag GTM 139, New York-Heidelberg-Berlin, 1993.

[B] K. S. Brown, *Cohomology of Groups*, Springer Verlag GTM 87, New York-Heidelberg-Berlin, 1994.

[BM] Bergau, and Mennicke

[BH] Birman, and H. Hilden,

[C1] F. R. Cohen, *On the hyperelliptic mapping class groups, $SO(3)$, and $Spin^c(3)$*, American J. Math., **115** (1993), 389–434.

[C2] F.R. Cohen, *On configuration spaces, their homology, and Lie Algebras,* Journal of Pure and Applied Algebra **100** (1995), 19-42.

[CG] F.R. Cohen, S. Gitler, (i) On loop spaces of configuration spaces, submitted to the Memoirs of the AMS, and (ii) Loop spaces of configuration spaces, braid-like groups, and knots, to appear in the Proceedings of the 1998 Barcelona Conference on Algebraic Topology.

[CLM] F.R. Cohen, T.J. Lada and J.P. May, *The homology of iterated loop spaces,* Lecture Notes in Math., vol. **533**, Springer-Verlag (1976).

[CS] F.R. Cohen, T. Sato, in preparation.

[CX] F.R. Cohen, M. Xicoténcatl, submitted.

[DC] D. C. Cohen, *Monodromy of fibre-type arrangements and orbit configuration spaces,* to appear in Forum Mathematicum.

[DCS] D. C. Cohen, Alexander I. Suciu, *Homology of iterated semidirect products of free groups*, J. Pure Appl. Algebra **126** (1998), pg 87-120.

[EE] C. Earle, and J. Eells, *A fibre bundle description of Teichmüller theory,* J. Differential Geometry **3**(1969), 19-43.

[FH1] E. Fadell and S. Husseini, *The space of loops on configuration spaces and the Majer-Terracini index*, Topol. Methods in Nonlinear Anal., Journal of the Julius Schauder Center, **11** (1998), 249–271.

[FH2] E. Fadell and S. Husseini, *Geometry and Topology of Configuration Spaces,* in preparation.

[FN] E. Fadell and L. Neuwirth, *Configuration spaces,* Math. Scand. **10** (1962), 119-126.

[FR] M. Falk, and R. Randell, *The lower central series of a fiber-type arrangement,* Invent. Math. **82** (1985), 77-88.

[H] R. Hain, *Infinitesimal presentations of the Torelli group,* J. Amer. Math. Soc. **10** (1997),597-691.

[K1] T. Kohno, *Linear represenations of braid groups and classical Yang-Baxter equations,* Cont. Math. **78** (1988), 339-363.

[K2] T. Kohno, *Vassiliev invariants and de Rham complex on the space of knots,* Cont. Math. **179** (1994), 123-138.

[K3] T. Kohno, *Elliptic KZ system, braid groups of the torus and Vassiliev invariants,* Topology and its Applications, **78** (1997), 79-94.

[KO] T. Kohno, and T. Oda, *The lower central series of the pure braid group of an algebraic curve,* Advanced Studies in Pure Mathematics, Galois Represntations and Arithmetic Algebraic Geometry **12** (1987), 201-219.

[MM] J. W. Milnor, and J.C. Moore, *On the structure of Hopf algebras,* Ann. of Math. **81** (1965), 211-264.

[R] Derek J.S. Robinson, *A Course in the Theory of Groups*, Springer Verlag GTM 80, New York-Heidelberg-Berlin, 1995.

[S] N. E. Steenrod, *The Topology of Fibre Bundles,* Princeton University Press.

[Sa] T. Sato, *On free groups and morphisms of coalgebras,* student topology seminar, Univ. of Roch., Spring 1998, preprint.

[Smale] S. Smale, P.A.M.S.

[Spa] Edwin H. Spanier, *Algebraic Topology*, Springer Verlag, New York-Heidelberg-Berlin, 1966.

[Wh] Whitehead, *Elements of Homotopy Theory*, Springer Verlag GTM 61, New York-Heidelberg-Berlin, .

[W] S. Willerton, thesis.

[X] M. Xicoténcatl, *Orbit configuration spaces, infinitesimal braid relations, and equivariant function spaces,* Ph.D. thesis, University of Rochester, Spring 1997.

[Xia] Yining Xia,