Contemporary Mathematics

A Quick Trip through Localization

Joseph A. Neisendorfer

Abstract. This is an expository paper on the localization of simply connected spaces. The form of this theory is due to Dror Farjoun and to Bousfield. It includes in one treatment the classical localization of inverting primes and that of $p-$completion. This paper presents these localizations in a geometric form for spaces and also in a closely related algebraic form for abelian groups. In addition, it includes the exotic localizations related to Miller's theorem. These exotic localizations give another proof of Serre's theorem that simply connected finite complexes are either contractible or have infinitely many nonzero homotopy groups. We also give a proof of Serre's conjecture that the homotopy groups of these spaces are either all zero or have infinitely many nonzero torsion components.

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Contents

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1. Geometric localization

The localization theory presented here is due independently to Bousfield [**2, 3**] and to Dror Farjoun [**8**]. Geometric localization involves the choice of some spaces to be local and the choice of some collection of maps to be local equivalences. Although we could have begun by choosing a map or even a whole set of maps to be local equivalences, we begin with the slightly restricted situation where the localization is the consequence of choosing one space to be locally equivalent to a point. This form of localization can also be called nullification.

Let M be a fixed connected pointed space and let X, Y, $\ldots$, A, B, $\ldots$ be simply connected pointed spaces. In this section, we describe the localization theory due independently to Bousfield [**2, 3**] and Dror Farjoun [**8**].

A space X is called $M-$**null or** $* \to M$ **local** if any of the following equivalent conditions hold:

a) The map of unbased mapping spaces

$$map(M, X) \to map(*, X) = X$$

is a weak equivalence, that is, $\pi_i map(M, X) \xrightarrow{\simeq} \pi_i X$ is an isomorphism for all $i \geq 0$ and all choices of basepoints.

b) The based mapping space $map_*(M, X)$ is weakly contractible, that is, the homotopy group $\pi_i map_*(M, X) = 0$ for all $i \geq 0$.

c) For all $i \geq 0$,

$$0 = \pi_i map_*(M, X) = [\Sigma^i M, X]_*$$

In this sense, a local space X is one which sees M as weakly contractible from the point of view of the based and unbased mapping spaces

$$map_*(M, X), \quad map(M, X).$$

Note: The equivalence of a) and b) is implied by the fibration sequence

$$map_*(M, X) \to map(M, X) \xrightarrow{eval} X$$

where $eval(f) = f(*)$ is the evaluation of a function on the basepoint.

A map $f : A \to B$ is a **local equivalence** if either of the following equivalent conditions hold:

a)

$$f^* : map_*(B, X) \to map_*(A, X)$$

is a weak equivalence for all local X.

b) For all local X and for all $i \geq 0$, the maps

$$\Sigma^i f^* : [\Sigma^i B, X]_* \to [\Sigma^i A, X]_*$$

are bijections.

c) For all local X and for all $i \geq 0$, the maps

$$f^* : [B, \Omega^i X]_* \to [A, \Omega^i X]_*$$

are bijections.

d)

$$f^* : map(B, X) \to map(A, X)$$

is a weak equivalence for all local X.

For example, the map $* \to M$ is a local equivalence.

A map $X \xrightarrow{\iota} LX = L_M X$ is called $M-$**localization** if the following two conditions are both satisfied

a) LX is local.

b) ι is a local equivalence.

We often refer to $LX = L_M X$ as the localization of X and omit mention of the map ι.

THEOREM 1.1. *For all connected spaces M and all simply connected spaces X, the localization $\iota : X \to LX$ exists, LX is simply connected, localization is unique up to homotopy, and localization is a functor on the homotopy category.*

REMARK 1.2. In fact, localization is represented by a strict functor LX, that is, by a functor on the category of simply connected pointed spaces and continuous pointed maps.

In the next section, we will prove that localization exists as a strict functor and is simply connected. Assume that we have done so.

The fact that localization is unique up to homotopy follows from the fact that localization is a functor on the homotopy category. In turn, the latter follows from the fact that, in the homotopy category, the localization $\iota : X \to LX$ is the universal map from X to a local space.

That is, for all local Y and all maps $f : X \to Y$, there is a unique homotopy class $g : LX \to Y$ such that $\iota^* g = g \cdot \iota \simeq f$.

Thus, given a map $h : X_1 \to X_2$ there is a unique homotopy class $Lh : LX_1 \to LX_2$ such that

$$\begin{array}{ccc} X_1 & \xrightarrow{h} & X_2 \\ \downarrow \iota & & \downarrow \iota \\ LX_1 & \xrightarrow{Lh} & LX_2 \end{array}$$

is homotopy commutative.

It is clear that Lh defines a functor on the homotopy category.

2. Geometric localization exists

We will prove that localization $LX = L_M X$ exists for all simply connected pointed spaces X. In order to avoid having to use sophisticated set theory, we will restrict our proof to the case where M is a connected pointed space which is a countable CW complex. In fact, this includes all useful examples that we know of.

Consider the Barratt-Puppe sequence

$$\bigvee \Sigma^j M \xrightarrow{F} X \to C_F \to \bigvee \Sigma^{j+1} M$$

where the bouquet is taken over all $j \geq 0$ and all maps $f : \Sigma^j M \to X$. The map F is the map defined by f on each summand. The van Kampen theorem implies that $C_F = X_+$ is simply connected if X is simply connected.

We denote the above sequence and its suspensions by

$$\Sigma^i M_\sigma \xrightarrow{\Sigma^i F} \Sigma^i X \to \Sigma^i X_+ \to \Sigma^{i+1} M_\sigma.$$

Since the Barratt-Puppe sequence and its suspensions are coexact, it follows that, for all local Y, there are bijections

$$[\Sigma^i X_+, Y]_* \xrightarrow{\simeq} [\Sigma^i X, Y]_*.$$

Therefore the map $X \to X_+$ is a local equivalence and all $\Sigma^i M \xrightarrow{f} X \to X_+$ are null homotopic.

We now construct the localization LX by transfinite recursion. Let α be an ordinal and set

$$\begin{array}{lllll} X_0 & = & X & & \\ X_\beta & = & (X_\alpha)_+ & & \text{whenever } \beta = \alpha+1 \text{ is a successor ordinal} \\ X_\beta & = & \lim_{\alpha<\beta} X_\alpha & = \bigcup_{\alpha<\beta} X_\alpha & \text{whenever } \beta = \text{limit ordinal} \\ LX & = & X_\Omega & & \text{where } \Omega = \text{the first uncountable ordinal} \end{array}$$

That is, if Ω is the first uncountable ordinal, the localization sits at the end of the process

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\omega \to X_{\omega+1} \to \dots$$
$$\to X_\alpha \to \cdots \to \lim_{\alpha<\Omega} X_\alpha = X_\Omega = LX.$$

We claim that $\iota : X \to LX$ is localization and simply connected. It is simply connected since it is a limit of simply connected spaces.

We need to check that ι is a local equivalence and that LX is local.

1) Since $X_\alpha \to X_{\alpha+1}$ is a direct system of cofibrations and $X_\beta = \lim_{\alpha<\beta} X_\alpha$ for limit ordinals β, it follows that $map_*(X_\alpha, Y) \leftarrow map_*(X_{\alpha+1}, Y)$ is an inverse system of fibrations and $map_*(X_\beta, Y) = \lim_{\alpha<\beta} map_*(X_\alpha, Y)$ for limit ordinals β. For all local Y and for all $i \geq 0$,

$$\pi_i map_*(X_\Omega, Y) = \pi_i map_*(\lim_{\rightarrow} X_\alpha, Y) = \lim_{\leftarrow} \pi_i map_*(X_\alpha, Y) \simeq \pi_i map_*(X, Y)$$

Thus, ι is a local equivalence.

2) Since M is a countable CW complex, M is an increasing union of finite subcomplexes $M_n \subseteq M$,

$$M = \bigcup_{n<\infty} M_n.$$

Let $f : \Sigma^i M \to X_\Omega$ be any map.

$$\forall n, \quad \exists \text{ an ordinal } \alpha(n) < \Omega$$

such that $f : \Sigma^i M_n \to X_{\alpha(n)}$.

Therefore,

$$\Sigma^i M \to \bigcup X_{\alpha(n)} = \lim_{\rightarrow} X_{\alpha(n)} = X_{\lim \alpha(n)} = X_\beta.$$

Since Ω is not a countable limit of lesser ordinals, it follows that $\beta < \Omega$ and that $\beta + 1 < \Omega$. Hence

$$\Sigma^i M \to X_\beta \to X_{\beta+1} \to X_\Omega$$

is null homotopic. Thus, $LX = L_M X = X_\Omega$ is $M-$null or local.

3. Localization of abelian groups

The localization theory or nullification theory of abelian groups is the consequence of a fixed group being declared to be locally equivalent to zero and where the role of the homotopy groups of the mapping space in the geometric theory is replaced by the Ext groups [**5, 12**] in the algebraic theory.

In this section, let $M =$ a fixed abelian group.

An abelian group X is $M-$**null or** $0 \to M$ **local** if certain Ext groups vanish, that is, if

$$0 = Ext^*(M, X) = \begin{cases} hom(M, X) & * = 0 \\ Ext(M, X) & * = 0 \end{cases}$$

REMARK 3.1. Thus, the algebraic analog of the geometric mapping space is $hom(P_*, X)$ where $P_* \to M$ is a free resolution of the abelian group M. The resulting derived functors are the homology groups of this complex, $H^* hom(P_*, X) = Ext^*(M, X)$, and it is well known that they are independent up to natural isomorphism of the choice of the projective resolution.

A homomorphism $f : A \to B$ of abelian groups is a **local equivalence** if, for all local X,

$$f^* : hom(B, X) \to hom(A, X)$$

is a bijection.

We shall say that f is a **strong local equivalence** if, for all local X,

$$f^* : Ext^*(B, X) \to Ext^*(A, X)$$

is a bijection.

A homomorphism $\iota : X \to L_M = LX$ is a localization if

1) ι is a local equivalence.

2) LX is local.

REMARK 3.2. Even though strong local equivalence is the straightforward algebraic analog to geometric local equivalence, that is, to the weak homotopy equivalence of mapping spaces, the above concept of local equivalence is the correct concept to use in the algebraic localization of abelian groups. In the case of the classical localization of abelian groups wherein primes are inverted, the two concepts of local equivalence and of strong local equivalence coincide. But, in the case where the localization is what is called $p-$completion, they are different. In this latter case, the localization $X \to LX$ may not be a strong local equivalence.

Exercise

a) If localization exists, then any homomorphism $f : X \to Y$ into a local abelian group Y has a unique extension to a map $g : LX \to Y$, that is, $g \cdot \iota = f$.

b) If it exists, localization is a functor and unique up to natural isomorphism.

In fact, Gustavo Granja has given an unpublished functorial construction of localization with respect to any fixed abelian group M. But we shall omit it since we are more interested in giving details of the two most important special cases, classical localization and completion.

4. Classical localization of abelian groups, inverting a set of primes

Inverting a set of primes is equivalent to the nullification of the set of corresponding cyclic groups, or equivalently, to the nullification of the direct sum of these cyclic groups.

Let $\mathcal{S}$ be a set of primes and let $M = M_{\mathcal{S}}$ be the abelian group

$$\bigoplus_{q \in \mathcal{S}} Z/qZ.$$

Inverting the primes in $\mathcal{S}$ means localization with respect to this abelian group $M = M_{\mathcal{S}}$.

Recall that an abelian group X is called $M-$ null or $0 \to M$ local if

$$0 = Ext^*(M, X), \quad * = 0, 1.$$

This is equivalent to $\forall q \in \mathcal{S}$

$$0 = hom(Z/qZ, X) = \{x \in X \quad | \quad qx = 0\} = Tor(Z/qZ, X)$$

and

$$0 = Ext(Z/qZ, X) = X/qX = X \otimes Z/qZ.$$

In other words, X is local if, $\forall q \in \mathcal{S}$, multiplication by q is an isomorphism

$$q : X \xrightarrow{\simeq} X.$$

That is, q is inverted in X.

Lemma 4.1. *The following are equivalent:*
a) X is $0 \to M$ local.
b) X is a module over the ring $Z[\mathcal{S}^{-1}]$
c) The natural map $X = X \otimes Z \xrightarrow{\simeq} X \otimes Z[\mathcal{S}^{-1}]$ is an isomorphism.

Recall that a homomorphism $f : A \to B$ is a local equivalence if, for all local X,

$$f^* : hom(B, X) \to hom(A, X)$$

is an isomorphism. In the case of classical localization of abelian groups, we have the following equivalent condition for a local equivalence.

Theorem 4.2. *$f : A \to B$ is a local equivalence if and only if*

$$f \otimes 1 : A \otimes Z[\mathcal{S}^{-1}] \to B \otimes Z[\mathcal{S}^{-1}]$$

is an isomorphism.

Proof. $f : A \to B$ is a local equivalence if and only if, for all local $X = X \otimes Z[\mathcal{S}^{-1}]$,

$$hom(B, X) \xrightarrow{\simeq} hom(A, X)$$

is an isomorphism. This is equivalent to

$$hom(B \otimes Z[\mathcal{S}^{-1}], X) \xrightarrow{\simeq} hom(A \otimes Z[\mathcal{S}^{-1}], X)$$

is an isomorphism. In turn, this is equivalent to

$$A \otimes Z[\mathcal{S}^{-1}] \xrightarrow{\simeq} B \otimes Z[\mathcal{S}^{-1}]$$

is an isomorphism.

1) The first equivalence is a consequence of the fact that

$$hom(C \otimes Z[\mathcal{S}^{-1}], X) \xrightarrow{\simeq} hom(C, X)$$

for all local X.

2) The backward implication of the second equivalence is clear. The forward implication is a consequence of:

Suppose

$$hom(B \otimes Z[\mathcal{S}^{-1}], X) \xrightarrow{\simeq} hom(A \otimes Z[\mathcal{S}^{-1}], X)$$

is an isomorphism for all local X.

Let $0 \to K \to A \otimes Z[\mathcal{S}^{-1}] \to B \otimes Z[\mathcal{S}^{-1}] \to C \to 0$ be exact. Then K and C are both local. Since

$$0 \leftarrow hom(C, C) \leftarrow hom(B \otimes Z[\mathcal{S}^{-1}], C) \leftarrow hom(A \otimes Z[\mathcal{S}^{-1}], C)$$

is exact, it follows that $C = 0$. Since

$$hom(B \otimes Z[\mathcal{S}^{-1}], A \otimes Z[\mathcal{S}^{-1}]) \leftarrow hom(A \otimes Z[\mathcal{S}^{-1}], A \otimes Z[\mathcal{S}^{-1}])$$

is an epimorphism, it follows that there is a map

$$g : B \otimes Z[\mathcal{S}^{-1}] \to A \otimes Z[\mathcal{S}^{-1}]$$

such that

$$g \cdot (f \otimes 1) = 1_{A \otimes Z[\mathcal{S}^{-1}]}.$$

Hence, $K = 0$. □

Localization exists and is given by $\iota : X \to X \otimes Z[\mathcal{S}^{-1}] = LX$. Clearly, LX is local and ι is a local equivalence.

EXAMPLE 4.3. If $\mathcal{S} = \{p\}$ = a single prime, then $Z[\mathcal{S}^{-1}] = Z[1/p]$ and this localization is called localization away from p.

EXAMPLE 4.4. If $\mathcal{S} = \mathcal{P}$ = the set of all primes, then $Z[\mathcal{S}^{-1}] = Q$ = the rational numbers and this localization is called rationalization.

EXAMPLE 4.5. If $\mathcal{S} = \mathcal{P} - \{p\}$ = the set of all primes but p, then $Z[\mathcal{S}^{-1}] = Z_{(p)}$ and this localization is called localization at p. We note that this gives the following localizations

$$LZ = Z \otimes Z_{(p)} = Z_{(p)}, \ LZ/p^r Z = Z/p^r Z \otimes Z_{(p)} = Z/p^r Z,$$

and, if $(n, p) = 1$,

$$LZ/nZ = Z/nZ \otimes Z_{(p)} = 0.$$

Exercise

In the case of classical localization show that any homomorphism of abelian groups which is a local equivalence is also a strong local equivalence.

Hint: Show that, if X is any local abelian group, then

$$Ext^*(A, X) = Ext^*(A \otimes Z[S^{-1}], X).$$

5. Classical localization of spaces

Classical localization of spaces is the nullification of the Moore spaces which are related to the set of primes to be inverted.

Let $\mathcal{S}$ = the set of primes to be inverted, $R = Z[\mathcal{S}^{-1}]$, and

$$G = M_{\mathcal{S}} = \bigoplus_{q\in\mathcal{S}} Z/qZ.$$

We describe a localization which inverts the primes in the homotopy groups (and the homology groups) of a simply connected space X.

Let $M = M(G,1)$ = the Moore space with nonzero first homology group isomorphic to G. The geometric localization that we seek is $M-$nullification or $* \to M$ localization.

For example, $M(Z/2Z,1) = RP^2 = S^1 \cup_2 e^2$ and, in general,

$$M(G,1) = \bigvee_{q\in\mathcal{S}} (S^1 \cup_q e^2).$$

Note that X is $M-$null if and only if X is $M(Z/qZ,1) = S^1 \cup_q e^2-$ null for all $q \in \mathcal{S}$. The Barratt-Puppe cofibration sequence and its suspensions

$$S^1 \xrightarrow{q} S^1 \to M(Z/qZ,1) \to S^2 \xrightarrow{q} S^2$$

$$S^{1+i} \xrightarrow{q} S^{1+i} \to \Sigma^i M(Z/qZ,1) \to S^{2+i} \xrightarrow{q} S^{2+i}$$

give the exact sequences

$$\pi_{1+i}X \xleftarrow{q} \pi_{1+i}X \leftarrow [\Sigma^i M(Z/qZ,1), X]_* \leftarrow \pi_{2+i}X \xleftarrow{q} \pi_{2+i}X$$

and hence the short exact sequences

$$0 \leftarrow ker\{\pi_{1+i}X \xrightarrow{q} \pi_{1+i}X\} \leftarrow [\Sigma^i M(Z/qZ,1), X]_* \leftarrow coker\{\pi_{2+i}X \xrightarrow{q} \pi_{2+i}X\} \leftarrow 0$$

$$0 \leftarrow hom(Z/qZ, \pi_{1+i}X) \leftarrow [\Sigma^i M(Z/qZ,1), X]_* \leftarrow Ext(Z/qZ, \pi_{2+i}X) \leftarrow 0.$$

Hence we have short exact sequences

$$0 \leftarrow hom(G, \pi_{1+i}X) \leftarrow [\Sigma^i M(G,1), X]_* \leftarrow Ext(G, \pi_{2+i}X) \leftarrow 0.$$

Remark 5.1. Since X is assumed to be simply connected, these short exact sequences are actually valid for any abelian group G provided we define the space $M(G,1)$ properly. Let $0 \to F_1 \to F_0 \to G$ be a choice of free resolution. If F is a free abelian group, then $M(F,1)$ is defined in the usual way as a bouquet of circles, one for each generator of F. Define $M(G,1)$ to be the cofibre of the map $M(F_1,1) \to M(F_0,1)$. We shall call $M(G,1)$ the first Moore space associated to the abelian group G.. The above short exact sequences are easy consequences of the cofibration sequences of the Moore spaces. The sets of homotopy classes $[\Sigma^i M(G,1), X]_*$ are well defined for all $i \geq 0$.

In the case we are considering, X is $M-$null or $M \to *$ local if and only if

$$[\Sigma^i M(G,1), X]_* = 0 \quad \forall i \geq 0.$$

This leads to the sequence of equivalences

$$\iff [\Sigma^i M(Z/qZ,1), X]_* = 0 \quad \forall i \geq 0 \text{ and } \forall q \in \mathcal{S}$$

$$\iff \pi_i X \quad \text{is uniquely } q \text{ divisible} \quad \forall i \geq 0 \text{ and } \forall q \in \mathcal{S}$$

$$\iff \pi_* X \xrightarrow{\simeq} \pi_* X \otimes R$$

$$\iff \pi_* X \;\; \text{is local, that is, } G- \text{ null.}$$

That is, a space is local if and only if all its homotopy groups are local.

Let $f : A \to B$ be a map of simply connected spaces. Then f is a local equivalence if and only if

$$\forall M - \text{null } Y, \quad map_*(A, Y) \xleftarrow{\simeq} map_*(B, Y)$$

is a weak equivalence, that is,

$$\forall M - \text{null} \;\; Y \text{ and } \;\forall i \geq 0, \quad [\Sigma^i A, Y]_* \xleftarrow{\simeq} [\Sigma^i B, Y]_*$$

is a bijection.

Without loss of generality, we can assume $A \subseteq B$ and, since the spaces A and B are simply connected, the above is equivalent to the vanishing of the obstructions in relative cohomology with untwisted coefficients [**24**] for all local Y :

$$H^*(\Sigma^i B, \Sigma^i A; \pi_* Y) = 0 \iff H^*(B, A; D) = 0$$

for all local $D \simeq D \otimes R$ and, by the universal coefficient theorem, this is equivalent to

$$H_*(B, A; R) = 0,$$

that is,

$$H_* A \otimes R \to H_* B \otimes R$$

is an isomorphism. **Hence, a map is a local equivalence if and only if it is a local homology isomorphism.**

In summary, if we set

$$M = \bigvee_{q \in \mathcal{S}} M(Z/qZ, 1), \quad R = Z[\mathcal{S}^{-1}],$$

then:

1) A simply connected X is $M-$null if and only if $\pi_* X$ is local if and only if $\pi_* X \simeq \pi_* X \otimes R$.

2) $A \to B$ is a local equivalence of simply connected spaces if and only if $H_* A \otimes R \xrightarrow{\simeq} H_* B \otimes R$ is a local homology isomorphism.

3) $\iota : X \to Y$ is localization if and only if $\pi_* Y$ is local and ι is a local homology equivalence.

We denote $LX = X \otimes R$.

Exercise

If $X \to LX$ is classical localization, then

$$\pi_* LX = \pi_* X \otimes R, \quad \overline{H}_* LX = \overline{H}_* X \otimes R.$$

Hint: First show that the above is true when X is a $K(\pi, 1)$ with π abelian and finitely generated, then for general abelian π. Then show that it is true when X is an Eilenberg-MacLane space. Then show that it is true when X has a finite Postnikov system. Then show that it is true for simply connected X.

6. Limits and derived functors

In order to discuss the concept of completion we need to recall some facts concerning inverse limits. These facts can be found in the book by Atiyah and McDonald [**1**].

A sequential inverse system of abelian groups $\{A_n, p_n\}$ is a collection of abelian groups A_n and homomorphisms $p_n : A_{n+1} \to A_n$ for $n \geq 1$. Morphisms of inverse systems $A_n \to B_n$ are defined in the obvious way by a commutative diagram of homomorphisms

$$\begin{array}{ccccccccc} A_1 & \leftarrow & A_2 & \leftarrow & A_3 & \leftarrow & A_4 & \leftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & . \\ B_1 & \leftarrow & B_2 & \leftarrow & B_3 & \leftarrow & B_4 & \leftarrow & \dots \end{array}$$

Given a sequential inverse system $\{A_n, p_n\}$, an inverse limit is an abelian group A together with morphisms $q_n : A \to A_n$ which are coherent in the sense that $p_n q_n = p_{n+1}$ for all $n \geq 1$. If B is another abelian group with coherent morphisms $r_n : B \to A_n$, then an inverse limit A is required to have the universal property that there always exists a unique morphism $r : B \to A$ such that $rp_n = r_n$ for all $n \geq 1$. This characterizes inverse limits uniquely up to isomorphism and we denote any inverse limit by

$$A = \lim_{\leftarrow} A_n.$$

The inverse limit and its derived functor can be computed as the cohomology of a certain cochain complex as follows:

Consider the cochain complex

$$0 \to \prod_n A_n \xrightarrow{\Phi} \prod_n A_n \to 0$$

where $\Phi(a_n) = (a_n - p_n(a_{n+1}))$. The cohomology of this complex defines the inverse limit functor and its derived functor:

The inverse limit functor is

$$\lim_{\leftarrow} A_n = ker(\Phi)$$

and the first derived functor of inverse limit is

$$\lim_{\leftarrow}{}^1 A_n = coker(\Phi).$$

Thus $(a_n) \in \lim_{\leftarrow} A_n$ if and only if $a_n = p_n(a_{n+1})$ for all $n \geq 1$.

Given a short exact sequence of inverse systems

$$0 \to A_n \to B_n \to C_n \to 0$$

we get a short exact sequence of cochain complexes and hence a long exact sequence of cohomology groups

$$0 \to \lim_{\leftarrow} A_n \to \lim_{\leftarrow} B_n \to \lim_{\leftarrow} C_n \xrightarrow{\delta} \lim_{\leftarrow}{}^1 A_n \to \lim_{\leftarrow}{}^1 B_n \to \lim_{\leftarrow}{}^1 C_n \to 0.$$

We shall say that an inverse system A_n is eventually zero if for all n there is some k so that the composition $p_n \circ p_{n+1} \circ \cdots \circ p_{n+k-1} \circ p_{n+k}$:

$$A_n \leftarrow A_{n+1} \leftarrow A_{n+2} \leftarrow \cdots \leftarrow A_{n+k+1}$$

is zero. We denote this composition by p^k.

Lemma 6.1. *If an inverse system of abelian groups A_n is eventually zero then*

$$\lim_{\leftarrow} A_n = \lim_{\leftarrow}{}^1 A_n = 0.$$

Proof. The inverse limit vanishes since $(a_n) \in \lim_{\leftarrow} A_n$ implies that

$$a_n = p_n(a_{n+1}) = p_n \circ p_{n+1}(a_{n+2}) = \cdots = p_n \circ p_{n+1} \circ \cdots \circ p_{n+k-1} \circ p_{n+k}(a_{n+k+1}) = 0$$

for all n.

To show that the derived functor is zero we need to show that the map $\Phi : \prod_n A_n \to \prod_n A_n$ is surjective. Let $(b_n) \in \prod_n A_n$ be any element. Since the system is eventually zero, the following infinite sums terminate and make sense:

$$\begin{aligned} a_1 &= b_1 + pb_2 + p^2 b_3 + p^3 b_4 + \ldots \\ a_2 &= b_2 + pb_3 + p^2 b_4 + p^3 b_5 + \ldots \\ a_3 &= b_3 + pb_4 + p^2 b_5 + p^3 b_6 + \ldots \\ &\ldots \end{aligned}$$

Then $\Phi(a_n) = (b_n)$ and Φ is surjective. □

For an inverse system A_n and $k \geq 0$, let $A_{n,k} = im(A_{n+k} \to \cdots \to A_n)$ be the $k-$th image inverse system and set

$$A_{n,\infty} = \bigcap_{k \geq 0} A_{n,k}$$

= the infinite image inverse system .

Definition 6.2. The inverse system A_n satisfies the Mittag-Leffler condition if the image inverse system converges in the sense that, for each n, there is a k such that $A_{n,k} = A_{n,\infty}$.

Since finite groups satisfy the descending chain condition, every inverse system of finite abelian groups satisfies the Mittag-Leffler condition.

An inverse system A_n is called epimorphic if every map $A_{n+1} \to A_n$ is an epimorphism. Clearly, an epimorphic inverse system satisfies the Mittag-Leffler condition. A simple exercise shows that

Lemma 6.3. *If A_n is an epimorphic inverse system, then*

$$\lim_{\leftarrow}{}^1 A_n = 0.$$

Theorem 6.4. *If A_n is an inverse system which satisfies the Mittag-Leffler condition then*

$$\lim_{\leftarrow}{}^1 A_n = 0.$$

Proof. Consider the short exact sequence of inverse systems

$$0 \to A_{n,\infty} \to A_n \to \frac{A_n}{A_{n,\infty}} \to 0.$$

The left hand system is epimorphic and the Mittag-Leffler condition implies that the right hand system is eventually zero. Since the derived functor vanishes for the systems on the ends, it vanishes for the middle system. □

Define a sequential direct system of abelian groups to be a collection of abelian groups A_n and homomorphisms $\iota_n : A_n \to A_{n+1}$ for all $n \geq 1$. Consider the complex

$$0 \to \bigoplus_n A_n \xrightarrow{\Psi} \bigoplus_n A_n \to 0$$

with $\Psi(a_n) = (a_n - \iota_n(a_n))$. Define the direct limit to be

$$\lim_{\rightarrow} A_n = coker(\Psi).$$

It is easy to see that Ψ is a monomorphism. Thus, the direct limit is the single nonvanishing homology group of this complex.

Hence, if $0 \to A_n \to B_n \to C_n \to 0$ is a short exact sequence of direct systems, then

$$0 \to \lim_{\rightarrow} A_n \to \lim_{\rightarrow} B_n \to \lim_{\rightarrow} C_n \to 0$$

is also a short exact sequence. In other words, the direct limit functor is exact.

7. Hom and Ext

In order to discuss completions of abelian groups, it is necessary to recall some connections between limits, hom, and the derived functors Ext.

Let A and B be abelian groups and consider the functor of two variables $hom(A, B)$. First we recall that an abelian group P is projective if and only if the covariant functor $hom(P, \quad)$ is exact. An abelian group Q is injective if and only if the contravariant functor $hom(\quad, Q)$ is exact. This leads to the fact that the derived functors of $hom(\quad,\quad)$ may be defined in two different ways:

1) If $P_* \to A \to 0$ is a projective resolution of A, then $Ext^q(A, B)$ is the $q-$the cohomology group of the cochain complex $hom(P_*, B)$. If $A_1 \to A_2 \to A_3 \to 0$ is exact, then

$$hom(A_1, B) \leftarrow hom(A_2, B) \leftarrow hom(A_3, B) \leftarrow 0$$

is exact. This shows that $Ext^0(A, B) = hom(A, B)$.

Of course, for abelian groups, projective resolutions can be chosen to have length ≤ 1 and thus $Ext^q(A, B) = 0$ if $q \geq 2$. In this case, we write $Ext^1(A, B) = Ext(A, B)$.

2) Alternatively, if $0 \to B \to Q_*$ is an injective resolution of B, then $Ext^q(A, B)$ is the $q-$the cohomology group of the cochain complex $hom(A, Q_*)$. The equivalence of this definition is an exercise. If $0 \to B_1 \to B_2 \to B_3$ is exact, then

$$0 \to hom(A, B_1) \to hom(A, B_2) \to hom(A, B_3)$$

is exact. This also shows that $Ext^0(A, B) = hom(A, B)$.

THEOREM 7.1. *If A_n is a direct system of abelian groups, then*

$$hom(\lim_{\rightarrow} A_n, B) = \lim_{\leftarrow} hom(A_n, B)$$

and there are short exact sequences

$$0 \to \lim_{\leftarrow}{}^1 hom(A_n, B) \to Ext(\lim_{\rightarrow} A_n, B) \to \lim_{\leftarrow} Ext(A_n, B) \to 0$$

PROOF. Apply the long exact exact sequence associated to hom and Ext to the short exact sequence $0 \to \oplus A_n \xrightarrow{\Psi} \oplus A_n \to \lim\limits_{\to} A_n \to 0$. □

We conclude this section with a result of Cartan-Eilenberg [**5**]. Suppose A, B, C are abelian groups, $P_* \to A \to 0$ is a projective resolution, and $0 \to C \to Q_*$ is an injective resolution. Consider the double complex

$$hom(P_* \otimes B, Q_*) = hom(P_*, hom(B, Q_*)).$$

If we note that the homology of $P_* \otimes B$ defines $Tor_*(A, B)$ and that the cohomology of $hom(B, Q_*)$ defines $Ext^*(B, C)$, we see that there are two spectral sequences converging to the cohomology of the associated total complex. To be specific:

1) If we filter the associated total complex by the injective resolution degree, we get a first quadrant spectral sequence with

$$E_1^{p,q} = hom(Tor_p(A, B), Q_q)$$
$$E_2^{p,q} = Ext^q(Tor_p(A, B), C)$$

and differentials $d_r : E_r^{p,q} \to E_r^{p+1-r,q+r}$

2) If we filter the associated total complex by the projective resolution degree, we get a first quadrant spectral sequence with

$$\overline{E}_1^{p,q} = hom(P_p, Ext^q(B, C))$$
$$\overline{E}_2^{p,q} = Ext^p(A, Ext^q(B, C))$$

and differentials $\overline{d}_r : \overline{E}_r^{p,q} \to \overline{E}_r^{p+r,q+1-r}$.

For abelian groups, the derived functors vanish beyond degree 1 and thus $E_2^{p,q} = E_\infty^{p,q}$, $\overline{E}_2^{p,q} = \overline{E}_\infty^{p,q}$. Hence,

THEOREM 7.2 (Cartan-Eilenberg). *For abelian groups,*

$$Ext^q(Tor_p(A, B), C) = 0 \textit{ for all } 1 \geq p, q \geq 0$$

if and only if

$$Ext^p(A, Ext^q(B, C)) = 0 \textit{ for all } 1 \geq p, q \geq 0.$$

8. $p-$completion of abelian groups

$p-$completions occur in the work of Bousfield and Kan [**4**]. We begin with the algebraic side to $p-$completion. Our treatment here is heavily influenced by the thesis of Shiffman [**22**].

Let p be a prime.

In this section, we answer 3 questions:

1) What does it mean for an abelian group X to be $p-$complete?

2) What does it mean for a homomorphism $f : A \to B$ of abelian groups to be a $p-$complete equivalence?

3) What is the $p-$completion of an abelian group X?

If X is finitely generated, we can answer question 3) right now. In this case, the $p-$completion is

$$\hat{X}_p = \lim_{\leftarrow} X/p^n X.$$

We shall see below that, in the general case, the $p-$completion is naturally isomorphic to the Ext group

$$Ext(Z(p^\infty), X).$$

We need the following:

a) the commutative ring:

$$Z[1/p] = \{\frac{x}{p^n} \,|n \geq 0,\, x \in Z\} = \{x\} \cup \{\frac{x}{p}\} \cup \{\frac{x}{p^2}\} \cup \cdots =$$

$$\lim_{\rightarrow} Z \xrightarrow{p} Z \xrightarrow{p} Z \xrightarrow{p} \dots$$

where the direct limit is mapped into $Z[1/p]$ by the maps

$$Z \to Z[1/p], \quad x \mapsto \frac{x}{p^n}.$$

b) the abelian group:

$$Z(p^\infty) = Z[1/p]/Z =$$

$$\lim_{\rightarrow} 0 \xrightarrow{\subseteq} Z/pZ \xrightarrow{\subseteq} Z/p^2Z \xrightarrow{\subseteq} \cdots =$$

the $p-$primary component of Q/Z.

Then $p-$completion of abelian groups is the nullification of the abelian group $Z[1/p]$, that is:

We define $p-$completion to be localization with respect to $0 \to Z[1/p]$. In particular, for X to be $p-$complete means that X is $Z[1/p]-$null or $0 \to Z[1/p]$ local, that is:

$$0 = Ext^*(Z[1/p], X) = \begin{cases} hom(Z[1/p], X), & * = 0 \\ Ext(Z[1/p], X), & * = 1 \end{cases}$$

$$= \begin{cases} hom(\lim_{\rightarrow} Z, X), & * = 0 \\ Ext(\lim_{\rightarrow} Z, X), & * = 1 \end{cases}$$

$$= \begin{cases} \lim_{\leftarrow} hom(Z, X), & * = 0 \\ \lim_{\leftarrow}{}^1 hom(Z, X), & * = 1 \end{cases}$$

$$= \begin{cases} \lim_{\leftarrow} X \xleftarrow{p} X \xleftarrow{p} X \xleftarrow{p} \dots, & * = 0 \\ \lim_{\leftarrow}{}^1 X \xleftarrow{p} X \xleftarrow{p} X \xleftarrow{p} \dots, & * = 1 \end{cases} .$$

We call the above sequence $X \xleftarrow{p} X \xleftarrow{p} \dots$ the $p-$sequence of X. The above equalities depend on the following two facts:

1)

$$hom(\lim_{\rightarrow} A_n, X) = \lim_{\leftarrow} hom(A_n, X)$$

and the short exact sequence

2)

$$0 \to \lim_{\leftarrow}{}^1 hom(A_n, X) \to Ext(\lim_{\rightarrow} A_n, X) \to \lim_{\leftarrow} Ext(A_n, X) \to 0.$$

Example 8.1. If $(n, p) = 1$, then the cyclic group Z/nZ is not $p-$complete since the fact that the $p-$sequence consists of isomorphisms implies that $\lim_{\leftarrow} \neq 0$.

EXAMPLE 8.2. The cyclic groups $X = Z/p^r Z$ are $p-$complete since:
There exists n such that $p^n = 0$ on X and thus

$$\lim_{\leftarrow} X = 0$$

and the Mittag-Leffler condition implies that

$$\lim_{\leftarrow}{}^1 X = 0$$

for the $p-$sequence.

EXAMPLE 8.3. The cyclic group Z is not $p-$complete since:
$\lim\limits_{\leftarrow} = 0$ but $\lim\limits_{\leftarrow}{}^1 = \lim\limits_{\leftarrow} Z/p^n Z \neq 0$ for the $p-$sequence. The latter follows from the short exact sequence of sequences

$$0 \to Z \xrightarrow{p^n} Z \to Z/p^n Z \to 0$$

where the first sequence is the $p-$sequence, the second sequence is the constant sequence, and the third sequence consists of the canonical epimorphisms $Z/p^{n+1}Z \to Z/p^n Z$.

A homomorphism $f : A \to B$ is a $p-$complete equivalence if:
for all $p-$complete X,

$$f^* : hom(B, X) \to hom(A, X)$$

is an isomorphism.

Recall that we have a stronger notion, f is called a strong $p-$complete equivalence if:
for all $p-$complete X

$$f^* : Ext^*(B, X) \to Ext^*(A, X), \quad * = 0, 1$$

is an isomorphism.

THEOREM 8.4. *If $ker\, f$ and $coker\, f$ are $Z[1/p]$ modules, then f is a strong $p-$complete equivalence.*

LEMMA 8.5. *If X is $p-$complete and M is a $Z[1/p]$ module, then*

$$Ext^*(M, X) = 0, \quad * = 0, 1.$$

PROOF. The lemma is clearly true if M is a free $Z[1/p]$ module. If $0 \to K \to F \to M \to 0$ is a short exact sequence of $Z[1/p]$ modules with F free, then the exactness of $0 \to hom(M, X) \to hom(F, X)$ implies that $hom(M, X) = 0$ and applying this result to K gives that

$$Ext(M, X) \simeq hom(K, X) = 0.$$

□

PROOF OF THEOREM 8.4. To prove the above theorem, consider the factorization of $f : A \to im\, f \to B$ and the short exact sequences

$$0 \to ker\, f \to A \to im\, f \to 0$$

$$0 \to im\, f \to B \to coker\, f \to 0.$$

The Lemma implies that

$$Ext^*(B, X) \xrightarrow{\simeq} Ext^*(im\, f, X) \xrightarrow{\simeq} Ext^*(A, X)$$

are isomorphisms. □

What is the $p-$completion of an abelian group X? The answer to this question is the map

$$\delta : X \to Ext(Z(p^\infty), X) = \hat{X}_p$$

below:

The short exact sequence $0 \to Z \to Z[1/p] \to Z(p^\infty) \to 0$ gives the long exact sequence

$$0 \to hom(Z(p^\infty), X) \to hom(Z[1/p], X) \to$$

$$hom(Z, X) = X \xrightarrow{\delta} Ext(Z(p^\infty), X) \to Ext(Z[1/p], X) \to 0.$$

We need to show that
1) $\hat{X}_p = Ext(Z(p^\infty), X)$ is $p-$complete and
2) the map $\delta : X \to \hat{X}_p$ is a $p-$complete equivalence.

Proof. To prove 1), we use the result of Cartan-Eilenberg in the previous section. Since Tor_* commutes with direct limits,

$$Tor_*(Z[1/p], Z(p^\infty)) = Tor_*(Z[1/p], \lim_{\to} Z/p^n Z)) = \lim_{\to} Tor_*(Z[1/p], Z/p^n Z) = 0$$

and the result of Cartan-Eilenberg implies that

$$Ext^*(Z[1/p], Ext^*(Z(p^\infty), X)) = 0$$

for all abelian groups X. □

Proof. To prove 2), factor the exact sequence

$$hom(Z[1/p], X) \to X \to Ext(Z(p^\infty), X) \to Ext(Z[1/p], X) \to 0$$

into the two exact sequences

$$hom(Z[1/p], X) \to X \xrightarrow{k} D \to 0$$

and

$$0 \to D \xrightarrow{j} Ext(Z(p^\infty), X) \to Ext(Z[1/p], X) \to 0.$$

Since multiplication by p is an isomorphism on $Z[1/p]$, it is an isomorphism on $Ext^*(Z[1/p], X)$, that is, $Ext^*(Z[1/p], X)$ is a $Z[1/p]$ module for all X. So the Lemma above shows that, for all $p-$complete Y,

$$Ext^*(Ext^*(Z[1/p], X), Y) = 0.$$

The exact sequences associated to hom and Ext show that the homomorphism j is a strong $p-$complete equivalence and that k is at least a $p-$complete equivalence. Hence, δ is a $p-$complete equivalence. □

We close this section with some information about the $p-$completion $\hat{X}_p = Ext(Z(p^\infty), X)$.

Since

$$Z(p^\infty) = \lim_{\rightarrow} Z/pZ \to Z/p^2Z \to Z/p^3Z \to \dots$$

there is a short exact sequence

$$0 \to \lim_{\leftarrow}{}^1 hom(Z/p^rZ, X) \to Ext(Z[1/p], X) \to \lim_{\leftarrow} Ext(Z/p^rZ, X) \to 0$$

where the righthand inverse limit is

$$\lim_{\leftarrow} Ext(Z/p^rZ, X) = \lim_{\leftarrow} X/p^rX$$

and the lefthand derived functor is

$$\lim_{\leftarrow}{}^1 hom(Z/p^rZ, X) = \lim_{\leftarrow}{}^1 (X_p \xleftarrow{p} X_{p^2} \xleftarrow{p} X_{p^3} \xleftarrow{p} \dots)$$

(where $X_{p^r} = \{x \in X \mid p^r x = 0\}$.)

If X is finitely generated, then the $p-$torsion in X has bounded order and the Mittag-Leffler condition implies that the above $\lim\limits_{\leftarrow}{}^1 = 0$. Hence

THEOREM 8.6. *If X is a finitely generated abelian group, there is an isomorphism*

$$\hat{X}_p \xrightarrow{\cong} \lim_{\leftarrow} X/p^rX.$$

EXAMPLE 8.7. Consider the abelian group $Z(p^\infty)$. We assert that its completion is $Z(p^\infty) \to 0$ and that this is therefore a local equivalence but it is not a strong local equivalence.

We show this in two steps:

1) $Ext(Z(p^\infty), Z(p^\infty))$ is uniquely $p-$divisible, that is, it is a $Z[1/p]$ module. Therefore, since it is $p-$ complete, it must be 0.

Consider the exact sequence

$$0 \to Z/pZ \to Z(p^\infty) \xrightarrow{p} Z(p^\infty) \to 0.$$

We get the long exact sequence [**5**]

$$0 \to hom(Z(p^\infty), Z/pZ) \to hom(Z(p^\infty), Z(p^\infty)) \xrightarrow{p} hom(Z(p^\infty), Z(p^\infty)) \to$$

$$Ext(Z(p^\infty), Z/pZ) \to Ext(Z(p^\infty), Z(p^\infty)) \xrightarrow{p} Ext(Z(p^\infty), Z(p^\infty)) \to 0.$$

Thus, $Ext(Z(p^\infty), Z(p^\infty))$ is $p-$divisible and we will know that it is uniquely $p-$divisible if we show both

$$Ext(Z(p^\infty), Z/pZ) = Z/pZ$$

and

$$hom(Z(p^\infty), Z(p^\infty)) = \hat{Z}_p = \lim_{\leftarrow} Z/p^rZ.$$

Since

$$Z(p^\infty) = \lim_{\rightarrow} Z/p^rZ,$$

we have an exact sequence

$$0 \to \lim_{\leftarrow}{}^1 hom(Z/p^rZ, Z/pZ) \to Ext(Z(p^\infty), Z/pZ) \to \lim_{\leftarrow} Ext(Z/p^r, Z/pZ) \to 0.$$

Since the maps $hom(Z/p^{r+1}Z, Z/pZ) \to hom(Z/p^rZ, Z/pZ)$ are all $Z/pZ \xrightarrow{p=0} Z/pZ$, we get that $\lim^1_{\leftarrow} hom(Z/p^rZ, Z/pZ) = 0$.

Since the maps $Ext(Z/p^{r+1}, Z/pZ) \to Ext(Z/p^r, Z/pZ)$ are all $Z/pZ \xrightarrow{=} Z/pZ$, we get that $\lim\limits_{\leftarrow} Ext(Z/p^rZ, Z/pZ) = Z/pZ$ and hence $Ext(Z(p^\infty), Z/pZ) = Z/pZ$.

We also get

$$\begin{aligned} hom(Z(p^\infty), Z(p^\infty)) &= hom(\lim_{\rightarrow} Z/p^rZ, Z(p^\infty)) = \lim_{\leftarrow} hom(Z/p^rZ, Z(p^\infty)) \\ &= \lim_{\leftarrow} Z/p^rZ = \hat{Z}_p. \end{aligned}$$

We pass onto the second step:

2) $Z(p^\infty) \to 0$ **is not a strong local equivalence, that is, there is a** $p-$**complete group, namely Z/pZ, such that**

$$Ext(Z(p^\infty), Z/pZ) \neq 0.$$

But we already have shown this in the proof of 1).

Exercise

If X is any abelian group, show that the $p-$completion $\hat{X}_p = 0$ if and only if X is $p-$divisible.

Hint: Use the long exact sequence associated to the short exact sequence

$$0 \to Z \to Z[1/p] \to Z(p^\infty) \to 0$$

and the short exact sequence associated to Ext of a direct limit.

9. $p-$completion of topological spaces

The simplest forms of localization arise from the nullification of Moore spaces. $p-$completion is one of these:

We define $p-$**completion of simply connected spaces as the** $M-$**nullification or** $* \to M$ **localization where** M **is the Moore space** $M(Z[1/p], 1)$ **with nonvanishing first homology group isomorphic to** $Z[1/p]$. **(See Remark 5.1 for the definition of** $M(Z[1/p], 1)$.**)**

A simply connected space X **is** $p-$**complete if and only if** X **is** $M-$**null, that is, if and only if**

$$\forall i \geq 0, \quad \pi_i map_*(M, X) = [\Sigma^i M, X]_* = 0.$$

Since we have the exact sequence

$$0 \leftarrow hom(Z[1/p], \pi_{i+1}X) \leftarrow [\Sigma^i M, X]_* \leftarrow Ext(Z[1/p], \pi_{i+2}X) \leftarrow 0,$$

it follows that X is $p-$complete if and only if

$$Ext^*(Z[1/p], \pi_i X) = 0 \quad \forall i \geq 0,$$

that is, if and only if all the homotopy groups $\pi_i X$ are $p-$complete.

Example 9.1. For $n \geq 2$, the Moore space $S^n \cup_{p^r} e^{n+1}$ is $p-$complete since all the homotopy groups are finite abelian $p-$torsion.

EXAMPLE 9.2. For $n \geq 2$ and $(s,p) = 1$, then $S^n \cup_s e^{n+1}$ is not $p-$complete.

EXAMPLE 9.3. For $n \geq 2$, the sphere S^n is not $p-$complete.

A map of simply connected spaces $f : A \to B$ is a $p-$complete equivalence if and only if:

$$\forall i \geq 0, \pi_i map_*(B,X) = [\Sigma^i B, X]_* \xrightarrow{\simeq} \pi_i map_*(A,X) = [\Sigma^i A, X]_*$$

is an isomorphism. Without loss of generality, we can assume that $A \subseteq B$ is an inclusion and, since the spaces are simply connected, obstruction theory applies with untwisted coefficients [**24**] and says that the above is equivalent to

$$H^*(B,A;D) = 0 \quad \forall p - \text{complete}\, D \Longleftrightarrow$$

$$Ext^j(H_*(B,A),D) = 0, \quad j = 0,1 \quad \forall p - \text{complete}\, D \Longleftrightarrow$$

by the Lemma below,

$$H_*(B,A) \text{ is a } Z[1/p] \text{ module} \Longleftrightarrow$$

$$H_*(B,A) \text{ is uniquely } p - \text{divisible.}$$

By the universal coefficient exact sequence

$$0 \to H_*(B,A) \otimes Z/pZ \to H_*(B,A;Z/pZ) \to Tor(H_{*-1}(B,A), Z/pZ) \to 0$$

this is equivalent to

$$H_*(B,A;Z/pZ) = 0$$

which is equivalent to

$$H_*(A;Z/pZ) \to H_*(B,Z/pZ)$$

is an isomorphism.

Hence, a $p-$complete equivalence is just a mod p-homology isomorphism.

LEMMA 9.4.

$$Ext^*(C,D) = 0 \quad \forall p - \textit{complete}\, D$$

if and only if the multiplication

$$p : C \to C$$

is an isomorphism.

PROOF. In the previous section, we showed that

$$Ext^*(C,D) = 0 \quad \forall p - \text{complete}\, D$$

if C is a $Z[1/p]$ module, which is the same as the backwards implication of the Lemma.

To prove the forwards implication of the Lemma, consider the exact sequence

$$0 \to ker\, p \to C \xrightarrow{p} C \to coker\, p \to 0.$$

where $ker\, p$ and $coker\, p$ are clearly $p-$complete.

The vanishing of $hom(C, coker\, p)$ immediately implies that $coker\, p = 0$ and hence the above sequence is in fact short exact.

Now the long exact sequence associated to the short exact sequence implies that

$$Ext^*(ker\, p, D) = 0 \quad \forall p - \text{complete}\, D$$

and hence that $ker\, p = 0$.

Hence,

$$Ext^*(C, D) = 0 \quad \forall p - \text{complete}\, D$$

implies that the multiplication

$$p : C \to C$$

is an isomorphism. □

Combining these characterizations of $p-$completeness and of $p-$complete equivalences gives that a map $\iota : X \to \hat{X}_p$ **is a $p-$completion if and only if the following 2 conditions are satisfied:**

1) The homotopy groups $\pi_* \hat{X}_p$ are $p-$complete.
2) The map ι is a mod p homology isomorphism.

Of course, $p-$completion satisfies the universal property:

$\forall f : X \to Y$ into a $p-$complete Y, there exists a map $\overline{f}; \hat{X}_p \to Y$ such that $\overline{f} \circ \iota \simeq f$ and this map $\overline{f}$ is unique up to homotopy.

Remark 9.5. If the homotopy groups $\pi_* X$ are finitely generated, then

$$\pi_* \hat{X}_p = (\pi_* X)\hat{}_p = \lim_{\leftarrow} \pi_* X / p^r \pi_* X.$$

This is a consequence of the fact that it is true for all Eilenberg-MacLane spaces with finitely generated abelian groups.

But the above remark can fail to be true when the homotopy groups are not finitely generated. To see this, consider the short exact sequence

$$0 \to Z \to Z[1/p] \to Z(p^\infty) \to 0$$

and the resulting fibration sequences

$$K(Z[1/p], i) \to K(Z(p^\infty), i) \to K(Z, i+1) \to K(Z[1/p], i+1).$$

For all $i \geq 1$, $K(Z[1/p], i)$ has 0 mod p homology. For $i = 1$, this follows from the direct limit

$$Z[1/p] = \lim_{\rightarrow} Z \xrightarrow{p} Z \xrightarrow{p} \dots$$

It follows for all $i \geq 1$ by induction and the mod p homology Serre spectral sequences of the path space fibrations $K(Z[1/p], i) \to PK(Z[1/p], i+1) \to K(Z[1/p], i+1)$.

Therefore, $K(Z(p^\infty), i) \to K(Z, i+1)$ is a mod p homology isomorphism.

Furthermore, $K(Z, i+1) \to K(\hat{Z}_p, i+1)$ is mod p homotopy isomorphism and therefore also a mod p homology isomorphism.

In conclusion, the composition

$$K(Z(p^\infty), i) \to K(Z, i+1) \to K(\hat{Z}_p, i+1)$$

is $p-$completion, that is,

$$K(Z(p^\infty), i)\hat{}_p = K(\hat{Z}_p, i+1).$$

In particular, the homotopy groups of the $p-$completion are not always the $p-$completion of the homotopy groups.

10. Miller's theorem and the Zabrodsky Lemma

Recall the theorem of Hopf [**11**]: **There exists a map** $\eta : S^3 \to S^2$ **such that** η **is not homotopic to a constant map. In fact,** $\pi_3 S^2 \neq 0$.

The more general theorem of Serre [**21**] is: **If** X **is a simply connected finite complex such that** $\pi_i X = 0$ **for** i **sufficiently large, then** X **is contractible.**

Our goal in this section is to prepare the way to give a new proof of Serre's theorem via Miller's theorem [**14**], Zabrodsky's lemma [**26**], and an exotic form of localization.

Recall without proof:

THEOREM 10.1 (Miller). *If* X *is simply connected and* $H_i(X; Z/pZ) = 0$ *for* i *sufficiently large, then*

$$\pi_i map_*(BZ/pZ, X) = 0 \quad \forall i \geq 0$$

or, in its equivalent form,

$$\pi_* map(BZ/pZ, X) \xrightarrow{\simeq} \pi_i X \quad \forall i \geq 0.$$

In the language of localization, such an X is $* \to BZ/pZ$ local or BZ/pZ-null. Hence, Miller's theorem is a localization theorem about a form of localization which is not defined by Moore spaces.

We shall prove the following result which is useful in the study of all localizations:

LEMMA 10.2 (Zabrodsky). *If* $F \to E \to B$ *is a fibre bundle sequence with* B *a CW complex and with connected fibre* F*, then*

$$\pi_i map(F, X) \simeq \pi_i X \quad \forall i \geq 0$$

implies that

$$\pi_i map(E, X) \xleftarrow{\simeq} \pi_i map(B, X) \quad \forall i \geq 0.$$

REMARK 10.3. Of course, this is equivalent to the following result for pointed mapping spaces

$$\pi_i map_*(F, X) \simeq 0 \quad \forall i \geq 0$$

implies that

$$\pi_i map_*(E, X) \xleftarrow{\simeq} \pi_i map_*(B, X) \quad \forall i \geq 0.$$

Note that this says that

$$map_*(B, X) \to map_*(E, X) \to map_*(F, X)$$

behaves as if it were a fibration.

PROOF. Zabrodsky's Lemma is proved as follows: We refer to the statement to be proved as $\mathcal{Z}$.

Given two towers of fibrations and a map from one to the other which is a weak homotopy equivalence at each level, the resulting map of inverse limits is a weak homotopy equivalence [**4, 6, 9**]. In other words, homotopy inverse limits are weakly homotopy invariant.

It follows that Zorn's Lemma implies that there is a maximal subcomplex C of B such that $\mathcal{Z}$ is true for the restriction of the fibre bundle to $\pi^{-1}(C) \to C$.

If $C \neq B$, then there exists a cell e such that $C \subsetneq C \cup e \subseteq B$.

By the usual trick of fattening, we can assume that e is a disk and $C \cap e = S$ is a sphere.

Consider the restrictions

a) $\pi^{-1}(C) \to C$.

b) $\pi^{-1}(e) = e \times F \to e$

c) $\pi^{-1}(S) = S \times F \to S$

where b) and c) must be trivial bundles since e is contractible.

We claim that $\mathcal{Z}$ is true for a), b), and c). Of course, a) is true by the choice of C,

And b) follows from

$$\pi_i map(e \times F, X) = \pi_i map(e, map(F, X)) = map(e, X) \quad \forall i \geq 0.$$

And c) is essentially the same as b).

We have a map of cofibration pushout diagrams

$$\begin{array}{ccc} \pi^{-1}(S) & \to & \pi^{-1}(C) \\ \downarrow & & \downarrow \\ \pi^{-1}(e) & \to & \pi^{-1}(C \cup e) \end{array} \xrightarrow{\pi} \begin{array}{ccc} S & \to & C \\ \downarrow & & \downarrow \\ e & \to & C \cup e \end{array}.$$

Applying the functor $map(\quad, X)$ to the above gives a map of fibration pullback diagrams

$$\begin{array}{ccc} map(\pi^{-1}(S), X) & \leftarrow & map(\pi^{-1}(C), X) \\ \uparrow & & \uparrow \\ map(\pi^{-1}(e), X) & \leftarrow & map(\pi^{-1}(C \cup e), X) \end{array} \xleftarrow{\pi^*} \begin{array}{ccc} map(S, X) & \leftarrow & map(C, X) \\ \uparrow & & \uparrow \\ map(e, X) & \leftarrow & map(C \cup e, X) \end{array}$$

Therefore, the Mayer-Vietoris homotopy sequence [**10**] of these pullback diagrams, the five lemma, and the fact that $\mathcal{Z}$ is true for a), b), and c) implies that $\mathcal{Z}$ is true for $\pi^{-1}(C \cup e) \to C \cup e$. Alternatively, one can use the fact that these pullback diagrams are homotopy inverse limits and therefore have the property of weak homotopy invariance.

Therefore, we must have $C = B$. $\square$

Remark 10.4. I am grateful to Emmanuel Dror-Farjoun for explaining the Mayer-Vietoris sequence to me in low dimensions and the sense in which it is exact.

Suppose that

$$\begin{array}{ccc} E & \xrightarrow{u} & X \\ \downarrow v & & \downarrow f \\ Y & \xrightarrow{g} & B \end{array}$$

is a homotopy pullback, that is, it is a pullback square with f and g being fibrations. The homotopy Mayer-Vietoris sequence [**10**] is, in dimensions $i \geq 1$, the long exact sequence

$$\cdots \to \pi_{i+1} B \xrightarrow{\partial} \pi_i E \xrightarrow{(u_*, v_*)} \pi_i X \oplus \pi_i Y \xrightarrow{f_* - g_*} \pi_i B \xrightarrow{\partial} \pi_{i-1} E \to \ldots.$$

In these dimensions, it is a long exact sequence of groups.

In the low dimensions, the homotopy Mayer-Vietoris sequence is

$$\begin{array}{ccccccccc}
\pi_1 X & & & & & & \pi_0 X & & \\
 & \searrow f_* & & & & u_* \nearrow & & \searrow f_* & \\
 & & \pi_1 B & \xrightarrow{\partial} & \pi_0 E & & & & \pi_0 B \\
 & \nearrow g_* & & & & v_* \searrow & & \nearrow g_* & \\
\pi_1 Y & & & & & & \pi_0 Y & &
\end{array}$$

and it is exact in the following way:

1) At $\pi_0 X$ and at $\pi_0 Y$, the exactness is: Given $\alpha \in \pi_0 X$ and $\beta \in \pi_0 Y$,,

$$f_*\alpha = g_*\beta \quad \text{if and only if} \quad \exists \gamma \in \pi_0 E \quad \text{such that} \quad u_*\gamma = \alpha, v_*\gamma = \beta.$$

2) At $\pi_0 E$, the exactness is: Given $\alpha \in \pi_0 E$, let $u_*\alpha = x$ and $v_*\alpha = y$, then

$$\exists \gamma \in \pi_1 B \text{ such that } \partial\gamma = \alpha.$$

(Here, the basepoint of the loops in B is the common image of the basepoints x in X and y in Y.)

3) At $\pi_1 B$, the exactness is: Given $\alpha, \beta \in \pi_1 B$,

$$\partial\alpha = \partial\beta \text{ if and only if } \exists \gamma \in \pi_1 X, \delta \in \pi_1 Y \text{ such that } \gamma * \alpha * \delta = (f_*\gamma)\alpha(g_*\delta) = \beta.$$

11. An exotic localization theorem and Serre's theorem

We now define exotic localization to be localization with respect to the map $* \to BZ/pZ \vee M(Z[1/p], 1)$. That is, exotic localization is nullification of both BZ/pZ and $M(Z[1/p], 1)$.

Hence, a simply connected space X is exotically local if and only if for all $i \geq 0$,

$$\pi_i map_*(BZ/pZ \vee M(Z[1/p], 1), X) = 0.$$

that is, X is both $p-$complete and $map_*(BZ/pZ, X)$ is weakly contractible.

It follows that, if $f : A \to B$ is a mod p homology isomorphism, then f is an exotic local equivalence: for all exotically local X,

$$f^* : map(B, X) \to map(A, X)$$

is a weak equivalence.

If X is a simply connected space, the $n-$connected covers of X are the sequence of spaces $X < n >$ and maps

$$\dots X < n+1 > \to X < n > \to \dots \to X < 3 > \to X < 2 > \to X < 1 > = X$$

where:

There are fibration sequences

$$K(\pi_{n+1} X, n) \to X < n+1 > \to X < n > \to K(\pi_{n+1}, n+1)$$

and

$$\pi_i(X < n >) = \begin{cases} 0, & i \leq n \\ \pi_i X, & i > n \end{cases} .$$

For example, $S^2 < 3 > = S^3$.

Note that the eventual vanishing of homotopy groups,

$$\pi_i X = 0 \quad \forall i > n$$

is equivalent to

$$\pi_i X < n > = 0 \quad \forall i \geq 0,$$

that is, $X < n >$ is weakly contractible.

We shall prove:

THEOREM 11.1 (Exotic localization theorem [**18**]). *If X is simply connected, if $H_i(X; Z/pZ) = 0$ for all sufficiently large i, and if $\pi_2 X$ is a torsion group, then*

$$\forall n, \quad L(X < n >) = \hat{X}_p$$

where

$$L = L_{BZ/pZ \vee M(Z[1/p],1)}$$

is exotic localization.

REMARK 11.2. The example $S^2 < 3 >= S^3$ shows that the hypothesis that $\pi_2 X$ is torsion is necessary. Joe Roitberg has coined the descriptive phrase $1\frac{1}{2}$-connected for this hypothesis.

The Exotic Localization Theorem has the following corollary:

THEOREM 11.3 (Serre [**21**]). *If Y is a simply connected finite complex and if $\pi_i Y = 0$ for i sufficiently large, then Y is contractible.*

PROOF. Let Y satisfy the hypotheses of the theorem. In particular, $Y < n >$ is contractible for sufficiently large n.

First of all, assume that $\pi_2 Y$ is finite. If p is any prime, the exotic localization theorem shows that

$$* \simeq LY < n > \simeq \hat{Y}_p.$$

Hence, Y is contractible in this case.

If $\pi_2 Y$ is not finite, write

$$\pi_2 Y = F \oplus G$$

where F is free abelian and G is finite. Since $CP^\infty = K(Z, 2)$, there is a bundle sequence

$$\Pi S^1 \to W \to Y \to \Pi CP^\infty$$

with ΠCP^∞ a notrivial product of complex projective spaces, $\pi_2 W = G =$ a finite group, and W is a simply connected finite dimensional CW complex. If we apply the previous case to W, we get that W is contractible, hence, the contradiction that

$$Y \simeq \Pi CP^\infty.$$

□

We conclude this section with a proof of the Exotic Localization Theorem:

Step 1: Let X be a simply connected space. Then $map_*(BZ/pZ, X) = 0$ is weakly contractible implies that

a1) For all finite groups $p-$groups G, $map_*(BG, X) = 0$ is weakly contractible, and

b1) For all locally finite $p-$groups G, $map_*(BG, X) = 0$ is weakly contractible.

PROOF. a1): If $G \neq e$ is a finite $p-$group, then there exists a subgroup

$$Z/pZ \simeq N \subseteq Z(G).$$

The exact sequence

$$0 \to N \to G \to G/N \to 0$$

gives the bundle sequence

$$BN \to BG \to B(G/N).$$

The Zabrodsky Lemma implies that there is a weak equivalence

$$map_*(BG, X) \xleftarrow{\simeq} map_*(B(G/N), X).$$

Induction on the order of G implies that $map_*(BG, X)$ is weakly contractible. □

PROOF. b1): If G is a locally finite $p-$group, then

$$G = \lim_{\rightarrow} G_\alpha$$

where G_α is a finite $p-$group. By a1),

$$map_*(BG, X) = \lim_{\leftarrow} map_*(BG_\alpha, X)$$

is weakly contractible. □

COROLLARY 11.4. *If G is an abelian $p-$torsion group and the space $map_*(BZ/pZ, X)$ is weakly contractible, then $map_*(BG, X)$ is weakly contractible.*

Step 2: If X is $p-$complete, if $map_*(BZ/pZ, X)$ is weakly contractible, and if G is torsion free abelian, then $map_*((K(G, 2), X)$ is weakly contractible.

PROOF. The exact sequence

$$0 \to Z \to Z[1/p] \to Z(p^\infty) \to 0$$

implies that the sequence

$$0 \to G \to Z[1/p] \otimes G \to Z(p^\infty) \otimes G \to 0$$

is exact. Hence, there is a fibration sequence

$$B(Z[1/p] \otimes G) \to B(Z(p^\infty) \otimes G) \to K(G, 2).$$

Since $B(Z[1/p] \otimes G)$ has trivial mod p homology, the Serre spectral sequence implies that

$$B(Z(p^\infty) \otimes G) \to K(G, 2)$$

is a mod p homology isomorphism, that is, a $p-$complete equivalence. Hence,

$$map_*(K(G, 2), X) \xrightarrow{\simeq} map_*(B(Z(p^\infty) \otimes G), X)$$

is weak equivalence and the above Corollary shows that both are weakly contractible. □

Step 3: If X is $p-$complete and H is an abelian torsion group with all elements of order relatively prime to p, then $map_*(BH, X)$ is weakly contractible.

PROOF. This is an immediate consequence of the fact that $BH \to *$ is a mod p homology isomorphism. □

Step 4: The following 3 implications are valid:

a4) If $map_*(G, X)$ is weakly contractible, then $map_*(BG, X)$ is weakly contractible.

b4) If there is an exact sequence $1 \to H \to T \to G \to 1$ and if both $map_*(BH, X)$ and $map_*(BG, X)$ are weakly contractible, then $map_*(BT, X)$ is weakly contractible.

c4) If both $map_*(BG, X)$ and $map_*(BH, X)$ are weakly contractible, then $map_*(B(G \times H), X)$ is weakly contractible.

PROOF. a4): Apply the Zabrodsky Lemma to the fibre bundle

$$G \to EG \to BG$$

with EG contractible. □

PROOF. b4): Apply the Zabrodsky Lemma to the fibre bundle

$$BG \to BT \to BH.$$

□

PROOF. c4): Clearly, b4) implies c4). □

Step 5: If X is $p-$complete, if $map_*(BZ/pZ, X)$ is weakly contractible, and if G is abelian, then $map_*(K(G, 2), X)$ is weakly contractible.

PROOF. Since there is an exact sequence $0 \to T \to G \to F \to 0$ with T torsion and F torsion free, this follows from the Corollary in Step 1, Step 2, Step 3, and Step 4. □

Step 6: If X is $p-$complete, if $map_*(BZ/pZ, X)$ is weakly contractible, and if π is an abelian group, then $map_*(K(\pi, n), X)$ is weakly contractible for all $n \geq 2$. If, in addition, π is a torsion abelian group, then the space $map_*(K(\pi, n), X)$ is weakly contractible for all $n \geq 1$.

PROOF. This follows by induction, using the above and the fact that $K(\pi, n) = BK(\pi, n-1)$. □

We now prove the Exotic Localization Theorem:

Step 7: If X is simply connected, if $H_i(X, Z/pZ) = 0$ for all sufficiently large i, and if $\pi_2 X$ is a torsion group, then

$$L(X < n >) = \hat{X}_p$$

where $L = L_{BZ/pZ \vee M(Z[1/p].1)}$ is the exotic localization.

PROOF. We claim that the composition

$$X < n > \xrightarrow{\iota} X \to \hat{X}_p$$

is the localization $L(X < n >)$.

1) First, $\iota : X < n > \to X < 1 > = X$ is an exotic local equivalence since:

Consider the fibre bundles

$$K(\pi_n(X), n) \to X < n > \to X < n-1 > .$$

Since $map_*(K(\pi_n(X), n), Y)$ is weakly contractible for all exotically local Y, the Zabrodsky Lemma implies that $X < n > \to X < n-1 >$ is an exotic local equivalence.

2) Since $X \to \hat{X}_p$ is a mod p homology isomorphism, it is an exotic local equivalence.

Hence, the composition

$$X < n > \xrightarrow{\iota} X \to \hat{X}_p$$

is an exotic local equivalence. But $\hat{X}_p$ is exotically local since it is $p-$complete and Miller's theorem implies that $map_*(BZ/pZ, \hat{X}_p)$ is weakly contractible. Therefore, the exotic localization is $L(X < n >) = \hat{X}_p$.

□

12. Applications of exotic localization

Exotic localization enables us to start with a $1\frac{1}{2}$ connected finite complex, to take a connected cover, and then to localize to recover the finite complex up to $p-$completion. This implies that not so much is lost when one takes the connected covers of such a complex. And there is a generalization of this result where the condition of being a finite complex is extended to the condition that the space be the iterated loop space of a finite complex.

These results have applications to H-spaces but, before we begin, we need to prove some lemmas show that localization preserves H-spaces.

Consider the localization L which inverts the map $* \to M$.

LEMMA 12.1. *If X and Y are simply connected and local, then $X \times Y$ is local.*

PROOF. This follows from $map(M, X \times Y) = map(M, X) \times map(M, Y)$ is weakly equivalent to $X \times Y$. □

LEMMA 12.2. *If X is simply connected and local and if C is an arbitrary connected space, then the unbased mapping space $map(C, X)$ is local.*

PROOF.

$$map(M, map(C, X)) = map(M \times C, X) = map(C, map(M, X))$$

is weakly equivalent to $map(C, X)$. □

LEMMA 12.3. *If X is simply connected and local and if C is an arbitrary connected space, then the based mapping space $map_*(C, X)$ is local.*

PROOF.

$$map_*(M, map_*(C, X)) = map_*(M \wedge C, X) = map_*(C, map_*(M, X))$$

is weakly contractible. □

For example, if X is local, then the loop space ΩX is local.

LEMMA 12.4. *If $A \to B$ is a local equivalence and C is an arbitrary space, then $A \times C \to B \times C$ is a local equivalence.*

PROOF. If X is local, then so is $map(C, X)$ and there is a weak equivalence

$$map(A \times C, X) = map(A, map(C, X)) \simeq map(B, map(C, X)) = map(B \times C, X).$$

□

COROLLARY 12.5. *If X and Y are simply connected, then*

$$L(X \times Y) = LX \times LY.$$

PROOF. By the first lemma above, $LX \times LY$ is local. By the last lemma above, $X \times Y \to (LX) \times Y \to (LX) \times (LY)$ is a composition of local equivalences and therefore a local equivalence. □

This has the corollary that localization preserves H-spaces, that is,

THEOREM 12.6. *If X is a simply connected H-space with multiplication $\mu : X \times X \to X$, then LX is an H-space with multiplication $L\mu : LX \times LX \to LX$. Furthermore, if $k : X \to X$ is a $k-$th power map, then $Lk : LX \to LX$ is a $k-$th power map.*

The exotic localization theorem gives:

THEOREM 12.7. *If X is a simply connected finite complex with $\pi_2 X$ finite and with the connected cover $X < n >$ an H-space for some n, then the $p-$completion $\hat{X}_p$ is an H-space.*

For example, $S^2 < 2 >= S^3$ is an H-space but $\hat{S^2}_p$ is not an H-space for any prime p. But, if $n > 1$, then $S^{2n} < k >$ is not an H-space for any k.

Note that localization at a prime and completion at a prime are the same when the homotopy groups are finite. That is, the natural properties of localization and completion give natural maps $X \to X_{(p)} \to \hat{X}_p$ and, if $\pi_i X$ is finite for all i, then localization and completion of homotopy groups is the same thing. Therefore $X_{(p)} \to \hat{X}_p$ is a homotopy equivalence.

For example, localization and completion of connected covers of odd dimensional spheres are homotopy equivalent,

$$S^{2n+1} < 2n+1 >_{(p)} \xrightarrow{\simeq} S^{2n+1} < 2n+1 >\hat{}_p.$$

We can use this to determine the number of loops needed to have a geometric exponent theorem.

THEOREM 12.8. *Localized at a prime $p > 2$, the H-space*

$$\Omega^{2n-1}(S^{2n+1} < 2n+1 >)$$

has a null homotopic p^n power map. But, localized at p, for all k the p^k-th power map on the H-space $\Omega^{2n-2}(S^{2n+1} < 2n+1 >)$ is not null homotopic.

PROOF. We begin with the proof of the first part: Let $S^{2n+1}\{p^r\}$ be the homotopy theoretic fibre of the p^r-th power map $S^{2n+1} \to S^{2n+1}$. In [**17**] it is shown that this is an H-space and that the p^r-th power map is null homotopic on $S^{2n+1}\{p^r\}$ and hence on all its loop spaces. In particular, the $p-$th power map is null homotopic on $\Omega S^{2p+1}\{p\}$. Selick [**19**] has shown that the loop space $\Omega(S^3 < 3 >)$ is a retract of $\Omega S^{2p+1}\{p\}$ via H-maps. Hence, the $p-$th power map is null homotopic on $\Omega(S^3 < 3 >)$.

The main result of [**7**] is that there is a map of spaces localized at $p > 2$

$$\pi : \Omega^2 S^{2n+1} \to S^{2n-1}$$

such that the composition with the double suspension

$$\Sigma^2 \circ \pi : \Omega^2 S^{2n+1} \to S^{2n-1} \to \Omega^2 S^{2n+1}$$

is the $p-$ power map, that is, $\Sigma^2 \circ \pi = p$.

Iterating this gives that, localized at $p > 2$, there is a factorization of the p^{n-1}-power map

$$\Omega^{2n-2} S^{2n+1} \to S^3 \to \Omega^{2n-2} S^{2n+1}.$$

Take connected covers and loop this one more time to get a factorization of the p^{n-1}-power map

$$\Omega^{2n-1}(S^{2n+1} < 2n+1 >) \to \Omega(S^3 < 3 >) \to \Omega^{2n-1}(S^{2n+1} < 2n+1 >).$$

Hence, Selick's result combined with [**7**] shows that $\Omega^{2n-1}(S^{2n+1} < 2n+1 >)$, localized or completed at $p > 2$, has a null homotopic p^n-th power map. [**20**] $\square$

PROOF. We continue with the proof of the second part of the Theorem:

On the other hand, localized or completed at any prime p, there is no p^k which is null homotopic on $\Omega^{2n-2}(S^{2n+1} < 2n+1 >)$.

In the exotic localization theorem, the condition that the mod p homology groups of X eventually vanish may be replaced by the condition that X be the iterated loops on such a space, provided that this iterated loop space is simply connected and has a torsion π_2. This follows from the fact that X local implies that the loop space ΩX is local. Thus the exotic localization theorem applies to $X = \Omega^{2n-2} S^{2n+1}$ since $\pi_2 X = \pi_{2n} S^{2n+1} = 0$ is finite. (It does not apply to $\Omega^{2n-1} S^{2n+1}$ since $\pi_2 = Z$ is not finite.)

Hence, if p^k were null homotopic on $\Omega^{2n-2}(S^{2n+1} < 2n+1 >)$, it would be null homotopic on the exotic localization

$$L(\Omega^{2n-2}(S^{2n+1} < 2n+1 >))$$

which is the $p-$completion of

$$\Omega^{2n-2} S^{2n+1}.$$

Since $\pi_3 \Omega^{2n-2} S^{2n+1} = \hat{Z}_p$ is torsion free, this is not true. $\square$

13. Serre's conjecture

In this section we prove the conjecture of Serre [**21**] which was first proved in [**13**]. Roughly speaking, his conjecture is that a noncontractible simply connected finite dimensional complex contains infinitely much torsion in its homotopy groups. This conjecture is a consequence of Miller's theorem [**14**] and therefore, although localization need not be mentioned, it is a consequence of a localization result.

Theorem 13.1 (Serre's conjecture). *If X is simply connected, the reduced mod p homology is nontrivial, $\overline{H}_*(X; Z/pZ) \neq 0$, and $H_i(X; Z/pZ) = 0$ for all sufficiently large i, then for infinity many i, the homotopy group $\pi_i(X)$ contains a subgroup of order p.*

Before we begin the proof of Serre's conjecture, we review the rationalization of simply connected spaces and Postnikov systems, $k-$invariants, and lifting problems.

Rationalization:

For a simply connected space Y, the rationalization of Y is the nullification of the Moore spaces $M(Z/qZ, 1)$ for all primes q, that is, it is the localization which inverts the map

$$* \to \bigvee_{\forall\, primes\, q} M(Z/qZ, 1).$$

If we denote this localization by $\iota : Y \to L_Q = Y \otimes Q$, then

$$\overline{H}_*(Y \otimes Q) = \overline{H}_*(Y) \otimes Q.$$

$$\pi_*(Y \otimes Q) = \pi_* Y \otimes Q.$$

Cartan-Serre proved a basic result on rationalization which is quoted in [**15**]:

Theorem 13.2. *If Y is a simply connected H-space, then*

$$X \otimes Q \simeq \Pi_{n \geq 2} K(\pi_n, n)$$

where $\pi_n = \pi_n Y \otimes Q$. They proved it in the strong form to be all explained below: all the rational k invariants are zero: $k_i(Y) \otimes Q = 0 \quad \forall i$.

Postnikov systems, $k-$invariants, and lifting problems:

If Y is a simply connected space or an H-space, then the Postnikov system [**23, 25**] represents Y as an inverse limit of fibrations $Y = \varprojlim Y_n$ with

$$\pi_i Y_n = \begin{cases} 0, & i > n \\ \pi_i Y, & i \leq n. \end{cases}$$

Thus, there are compatible maps $Y \to Y_n$ and a sequence of fibre maps

$$\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_3 \to Y_2 \to Y_1$$

such that

$$Y_1 = K(\pi_1 Y, 1)$$

and there are fibration sequences up to homotopy

$$K(\pi_n Y, n) \to Y_n \to Y_{n-1} \xrightarrow{k_n} K(\pi_n Y, n+1).$$

The $k-$invariant k_n is identified with the cohomology class $k_n = k_n(Y) \in H^{n+1}(Y;\pi_n Y)$.

Of course, the vanishing of the $k-$invariant, $0 = k_n$, implies that

$$Y_n \simeq Y_{n-1} \times K(\pi_n Y, n).$$

More generally, the lifting problem

$$\begin{array}{ccc} & & W \\ & f \swarrow & \downarrow f_2 \\ Y & \rightarrow & Y_2 \end{array}$$

has a solution f if and only if, for all n, the lifting problems

$$\begin{array}{cccc} & & W & \\ & f_n \swarrow & \downarrow f_{n-1} & \\ Y_n & \rightarrow & Y_{n-1} & \xrightarrow{k_n} K(\pi_n Y, n+1) \end{array}$$

have solutions f_n. This is equivalent to, for all n,

$$k_n \circ f_{n-1} = 0 \in H^{n+1}(W;\pi_n Y).$$

LEMMA 13.3 (Lifting Lemma). *If all spaces are localized at a prime p, the above lift exists if $H_*(W;Z)$ is a free $Z_{(p)}-$module, $\pi_* Y$ is torsion free, and Y is a simply connected H-space.*

PROOF. The obstructions to a lift are in $H^{n+1}(W;\pi_n Y) = hom(H_{n+1}W;\pi_n Y)$ which is contained in $hom(H_{n+1}W;\pi_n Y\otimes Q) = H^{n+1}(W;\pi_n Y \otimes Q)$ since $\pi_n Y$ is torsion free and the equalities are valid since $H_{n+1}W$ is free.

Under the above injective maps, the obstructions

$$k_n \circ f_{n-1} \mapsto (k_n \otimes Q)\circ f_{n-1}$$

and the latter is zero by the result of Cartan-Serre.

Hence, the obstructions are zero and the lift exists. □

PROOF. This is the proof of Serre's conjecture.

For simply connected Y and $n \geq 2$, the mod p homotopy groups [**16**] are defined by

$$\pi_n(Y;Z/pZ) = [S^{n-1}\cup_p e^n, Y]_*$$

and the cofibration sequence

$$S^{n-1} \xrightarrow{p} S^{n-1} \rightarrow S^{n-1}\cup_p e^n \rightarrow S^n \xrightarrow{p} S^n$$

yields the universal coefficient exact sequence

$$0 \rightarrow \pi_n Y \otimes Z/pZ \rightarrow \pi_n(Y;Z/pZ) \rightarrow Tor(\pi_{n-1}Y, Z/pZ) \rightarrow 0.$$

We begin this proof by showing that the mod p homotopy groups of X cannot eventually vanish, that is, there is no k such that $\pi_i(X;Z/pZ) = 0$ for all $i > k$. (The universal coefficient theorem implies that this is another proof of Serre's theorem.)

If the mod p homotopy groups of X eventually vanish, then the nontriviality of the mod p homology implies that is a nonzero mod p homotopy group

$$\pi_m(X;Z/pZ) \neq 0$$

with $m \geq 2$ as large as possible. The universal coefficient theorem implies that there are two possibilities:

1) $\pi_m X \otimes Z/pZ \neq 0$ or

2) $\pi_m X \otimes Z/pZ = 0$ and $Tor(\pi_{m-1}X, Z/pZ) \neq 0$. In this case, $m \geq 3$ since X is simply connected.

And, in both cases, $\pi_i(X; Z/pZ) = 0$ for all $i > m$. Hence, $\pi_i X$ is uniquely $p-$divisible if $i > m$ in both cases and also for $i = m$ in case 2). In other words, $\pi_i X$ is a $Z[1/p]$ module if $i > m$ in both cases and also for $i = m$ in case 2). In cases 1) and 2), we have that $\pi_m X$ has no nontrivial elements of order p.

Miller's theorem asserts that, for the X we are considering, $map_*(BZ/pZ, X)$ is weakly contractible and hence $map_*(BZ/pZ, \Omega^j X) = \Omega^j map_*(BZ/pZ, X)$ is weakly contractible for any iterated loop space.

Suppose that X satisfies case 2) and let $Y = (\Omega^{m-2}X)_0 =$ the component of the basepoint in the iterated loop space. There is a nontrivial map $f_1 : BZ/pZ \to K(\pi_{m-1}X, 1)$ and the obstructions to the lifting this up the Postnikov system to $f : BZ/pZ \to (\Omega^{m-1}X)_0$ are zero, that is,

$$k_i \in H^{i+1}(BZ/pZ, \pi_i Y) = 0, \quad i \geq 2$$

since $\pi_i Y = \pi_{i+m-2}X, \quad i+m-2 \geq m$, is a $Z[1/p]$ module. Hence, the composition $BZ/pZ \xrightarrow{f} Y \subseteq \Omega^{m-2}X$ is not null homotopic. This contradicts Miller's theorem.

Suppose that X satisfies case 1) and let $Y = \Omega^{m-2}X < 1 >=$ the universal cover of the iterated loop space. We have a monomorphism $\gamma : Z_{(p)} \to \pi_m X$ such that $Z_{(p)} \otimes Z/pZ \to \pi_m X \otimes Z/pZ$ is also a monomorphism. Hence, the composition

$$f_2 : BZ/pZ \xrightarrow{\alpha} K(Z_{(p)}, 2) \xrightarrow{\gamma} K(\pi_m X, 2)$$

is not null homotopic where γ is the induced map and α represents a generator of

$$H^2(BZ/pZ, Z_{(p)}) = Ext(Z/pZ, Z_{(p)}) = Z/pZ.$$

The map f_2 is not null homotopic since:

It is clear that

$$H^2(K(\pi_m X, 2), Z/pZ) = hom(\pi_m X, Z/pZ) = hom(\pi_m X \otimes Z/pZ, Z/pZ) \to$$
$$H^2(K(Z_{(p)}, 2), Z/pZ) = hom(Z_{(p)}, Z/pZ) = hom(Z_{(p)} \otimes Z/pZ, Z/pZ)$$

is an epimorphism. Therefore, the composition $f_2 = \gamma \circ \alpha$ induces the nonzero map

$$hom(\pi_m X \otimes Z/pZ, Z/pZ) \to hom(Z_{(p)} \otimes Z/pZ, Z/pZ) \xrightarrow{\simeq} Z/pZ$$

in the second mod p cohomology group.

The obstructions to lifting f_2 to a map $f : BZ/pZ \to \Omega^{m-2}X < 1 >$ are zero, that is,

$$k_i \in H^{i+1}(BZ/pZ, \pi_i Y) = 0, \quad i \geq 3$$

since $\pi_i Y = \pi_{i+m-2}X, \quad i + m - 2 \geq m + 1$, is a $Z[1/p]$ module. The composition $BZ/pZ \xrightarrow{f} Y \to \Omega^{m-2}X$ is not null homotopic since Y is a covering, f is not null homotopic and BZ/pZ is connected. This contradicts Miller's theorem.

Thus, we have shown that the mod p homotopy groups of X cannot eventually vanish, that is, there is no k such that $\pi_i(X; Z/pZ) = 0$ for all $i > k$.

Now suppose that $Tor(\pi_m X, Z/pZ) \neq 0$ for only finitely many m. Choose m sufficiently large that $Tor(\pi_i X, Z/pZ) = 0$ for all $i \geq m$ and

$$\pi_m(X; Z/pZ) = \pi_m X \otimes Z/pZ \neq 0.$$

As above, let $Y = \Omega^{m-2}X < 1 >$ and let $\gamma : K(Z_{(p)}, 2) \to K(\pi_m X, 2)$ be a map such that the composition

$$BZ/pZ \xrightarrow{\alpha} K(Z_{(p)}, 2) \xrightarrow{\gamma} K(\pi_m X, 2)$$

is not null homotopic. The previous Lifting Lemma implies that the obstructions to lifting γ are zero. As before, there is a composition

$$BZ/pZ \to Y \to \Omega^{m-2}X$$

which is not null homotopic and this contradicts Miller's theorem.

Serre's conjecture is proved. □

References

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley, 1969.

[2] A. K. Bousfield. Localization and periodicity in unstable homotopy theory. *Jour. Amer. Math. Soc.*, 7:831–874, 1994.

[3] A. K. Bousfield. Unstable localizations and periodicity. In C. Brota, C. Casacuberta, and G. Mislin, editors, *Algebraic Topology: New Trends in Localization and Periodicity.* Birkhäuser, 1996.

[4] A. K. Bousfield and D. M. Kan. *Homotopy Limits, Completions, and Localization, Lecture Notes in Math 304*. Springer-Verlag, 1972.

[5] H. Cartan and S. Eilenberg. *Homological Algebra*. Princeton University Press, 1956.

[6] W. Chachólski and J. Scherer. Homotopy theory of diagrams. *Mem. Amer. Math. Soc.*, 155(736):x+90, 2002.

[7] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer. The double suspension and exponents of the homotopy groups of spheres. *Ann. of Math.*, 110:549–565, 1979.

[8] E. Dror Farjoun. *Cellular Spaces, Null Spaces, and Homotopy Localization, Lecture Notes in Math. 1622*. Springer-Verlag, 1995.

[9] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith. *Homotopy Limit Functors on Model Categories and Homotopical Categories*. Amer. Math. Soc., 2004.

[10] E. Dyer and J. Roitberg. Note on sequences of Mayer-Vietoris type. *Proc. Amer. Math. Soc.*, 80:660–662, 1980.

[11] H. Hopf. Uber die abbildungen von spharen niedriger dimensionen. *Fund. Math.*, 25:427–440, 1935.

[12] S. MacLane. *Homology*. Springer-Verlag, 1963.

[13] C. A. McGibbon and J. A. Neisendorfer. On the homotopy groups of a finite dimensional space. *Comment. Math. Helv.*, 59:253–257, 1984.

[14] H. R. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math.*, 120:39–87, 1984.

[15] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math.*, 81:211–264, 1965.

[16] J. A. Neisendorfer. *Primary homotopy theory, Memoirs A.M.S. 232*. Amer. Math. Soc., 1980.

[17] J. A. Neisendorfer. Properties of certain H-spaces. *Quart. Jour. Math. Oxford*, 34:201–209, 1981.

[18] J. A. Neisendorfer. Localization and connected covers of finite complexes. *Contemp. Math.*, 181:385–390, 1995.

[19] P. S. Selick. Odd primary torsion in $\pi_k(S^3)$. *Topology*, 17:407–412, 1978.

[20] P. S. Selick. Space exponents for loop spaces of spheres. In W. Dwyer et al., editor, *Stable and Unstable Homotopy*. Amer. Math. Soc., 1998.

[21] J.-P. Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–231, 1953.

[22] S. J. Shiffman. *Ext p-completion in the homotopy category*. PhD thesis, Dartmouth College, 1974.

[23] E. H. Spanier. *Algebraic Topology*. McGraw-Hill, 1966.

[24] N. E. Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[25] G. W. Whitehead. *Elements of Homotopy Theory*. Springer-Verlag, 1978.

[26] A. Zabrodsky. Phantom maps and a theorem of H. Miller. *Israel J. Math.*, 58:129–143, 1987.

Department of Mathematics, University of Rochester, Rochester, NY 14625
E-mail address: jnei@math.rochester.edu