Basic inequalities:
\[ a_i \in \mathbb{R}, \quad b_i \in \mathbb{R} \]

Cauchy-Schwarz:
\[
| \sum_{i} a_i b_i | \leq \left( \sum_{i} a_i^2 \right)^{\frac{1}{2}} \left( \sum_{i} b_i^2 \right)^{\frac{1}{2}} \tag{\star}
\]

Let \( A = \left( \sum_{i} a_i^2 \right)^{\frac{1}{2}} \quad B = \left( \sum_{i} b_i^2 \right)^{\frac{1}{2}} \)

\( \star \) becomes
\[
\sum_{i} \frac{a_i}{A} \frac{b_i}{B} \leq 1
\]

Observe that
\[
\sum_{i} \left( \frac{a_i}{A} \right)^2 = \frac{1}{A^2} \sum_{i} a_i^2 = 1
\]

Similarly, \( \sum_{i} \left( \frac{b_i}{B} \right)^2 = 1 \)

We have thus reduced matters to showing that
\[
\sum_{i} a_i b_i \leq 1 \quad \text{if} \quad \sum_{i} a_i^2 = \sum_{i} b_i^2 = 1.
\]
Observe that

\[ a_i b_i \leq \frac{a_i^2 + b_i^2}{2} \quad \text{since} \]

\[ 0 \leq (a_i - b_i)^2 = a_i^2 + b_i^2 - 2a_i b_i \]

It follows that

\[ \sum a_i b_i \leq \frac{1}{2} \sum a_i^2 + \frac{1}{2} \sum b_i^2 \]

\[ = \frac{1}{2} + \frac{1}{2} = 1. \]

The following example illustrates the power of Cauchy-Schwarz:

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots
\end{pmatrix}
\]

not allowed!

no rectangles w/ 1's at the vertices.
More precisely, let $\{a_{ij}\}_{i,j=1}^n$ be an $n \times n$ matrix, where $a_{ij} = 0$ or 1. The "rectangle" condition means that if $j \neq j'$, then $a_{ij} \cdot a_{ij'} = 1$ for at most one value of $i$.

Question: How many 1's can our matrix possibly have?

\[
\left( \sum_{i=1}^n \sum_{j=1}^n a_{ij} \right)^2 = \left( \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \right) \cdot 1 \right)^2 \\
\leq \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \right)^2 \cdot \sum_{i=1}^n 1^2 \\
= n \sum_{i=1}^n \sum_{j=1}^n a_{ij} \cdot a_{ij'} \\
= n \sum_{i=1}^n \sum_{j=1}^n a_{ij} \cdot a_{ij'} \\
\text{In order to use our condition, we must have } a_{ij} \cdot a_{ij'} = 1 \text{ for at most one value of } i, j \neq j'$.

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} q_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} a_{ij} q_{ij} \\
= I + II \\
\leq n^3
\]

To estimate II, note that since \( j \neq i \), there is at most one \( i \) such that \( q_{ij} \cdot q_{ij} = 1 \). It follows that \( II \leq n \cdot n = n \cdot (n-1) \leq n^3 \).

We conclude that
\[
\left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \right) \leq 2n^3, \quad \text{so} \\
\sum_{ij=1}^{n} a_{ij} \leq \sqrt{2} n^{3/2}
\]
Exercise: For a sequence of $n \to \infty$, construct an $n \times n$ matrix satisfying the rectangle condition and containing $\sim n^2$ 1's.

Where does the rectangle condition come from?

n points, n lines

incidence = (point, line) ∈ points x line

I = total number of incidences
\( p_i, p_2, \cdots, p_n \) points

\( l_i \)

\( l_{ij} \)

\( l_{ij}' \)

\( 1 \) if \( i \)th point lies

on the \( j \)th line.

lines

\( p_i, p_i' \) not possible

because two points
determine a line!
Holder's inequality: \( 1 < p, q < \infty \) \( \frac{1}{p} + \frac{1}{q} = 1 \)

\( a_i, b_i \in \mathbb{R} \)

Then \( \left| \sum a_i b_i \right| \leq \left( \sum |a_i|^p \right)^{\frac{1}{p}} \left( \sum |b_i|^q \right)^{\frac{1}{q}} \)

**proof:**

Applying the same idea as in the proof of Cauchy–Schwarz, we divide both sides by \( AB \) and reduce matters to proving that \( \left| \sum a_i b_i \right| \leq 1 \) if \( A = B = 1 \)

What is the analog of the inequality \( ab \leq \frac{a^2 + b^2}{2} \) we used for Cauchy–Schwarz?

If we were to mimic the proof of C–S, we would need the inequality \( ab \leq a^p + b^q \), \( a, b \geq 0 \) \( \frac{1}{p} + \frac{1}{q} = 1 \)

But is it true?
The fact that \( \frac{1}{p} + \frac{1}{q} = 1 \) gives us a clue that convexity is involved, so we rewrite
\[
ab \leq \frac{a^p + b^q}{p + q}
\]
ab \leq (1 - t) a^p + t b^q

Let \( a = e^{(1-t)x} \) \( b = e^{ty} \) for some \( x, y \)

Then the inequality takes the form
\[
e^{(1-t)x + ty} \leq (1 - t) e^x + t e^y,
\]

which is just a statement that the exponential function is convex!

How do we really know that the exponential function is convex? We shall address this point comprehensively in a moment.
Let us first complete the proof of Hölder.

We must show that

\[ \left| \sum a_i b_i \right| \leq 1, \quad \left( \sum |a_i|^p \right)^{\frac{1}{p}} \left( \sum |b_i|^q \right)^{\frac{1}{q}} = 1 \]

\[ \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < \infty \]

We have

\[ \sum a_i b_i \leq \frac{1}{p} \sum |a_i|^p + \frac{1}{q} \sum |b_i|^q \]

\[ = \frac{1}{p} + \frac{1}{q} = 1 \]

Let us now explore convexity a bit more.

Let \( \varphi \) be a twice differentiable function such that \( \varphi''(x) \geq 0 \).

Is it true that \( x < y \)

\[ \varphi((1-z)x + zy) \leq (1-z)\varphi(x) + z\varphi(y) ? \]
Let
\[ F(t) = (1-t) \varphi(x) + tz \varphi(y) - \varphi((1-t)x + ty) \]
\[ F'(t) = \varphi(y) - \varphi(x) - (y-x) \varphi'(t) \varphi((1-t)x + ty) \]
\[ F''(t) = -(y-x)^2 \varphi''((1-t)x + ty) \leq 0 \]

\[ F(0) = F(1) = 0 \quad F''(t) \leq 0 \]

Does this imply that \( F \geq 0 \)?

\( \checkmark \) which is what we want

Suppose not! Then \( \exists \, t_0 \in (0,1) \) \( F(t_0) < 0 \),

By Mean Value Theorem, \( \exists \, \xi_0 \in (0,t_0) \) \( F'(\xi_0) \)

\( F(0) - F(0) = (\xi_0 - 0) \varphi'(\xi_0) \)

\( \text{negative} \quad \Rightarrow \quad F'(\xi_0) < 0 \)
Similarly, \( \exists c_0 \in (a, b) \)

\[
F'(a) - F'(b) = (1 - \epsilon_0) F'(c_0)
\]

0 positive \( \implies F'(c_0) > 0 \)

Applying MVT yet again, \( \exists c_1 \in (c_0, b) \)

\[
F(c_0) - F(b) = (c_0 - c_1) F''(c_1)
\]

\( \implies F''(c_1) > 0 \)

**CONTRADICTION!**