1

Integration

P_i = partition of $[a_i, b_i]$

$\hookrightarrow P = (P_1, P_2, \dots, P_n)$ = partition of $[a_1, b_1] \times \dots \times [a_n, b_n]$

$S \in P$ subrectangle

$m_S(f) = \inf\{f(x) : x \in S\}$

$M_S(f) = \sup\{f(x) : x \in S\}$

$v(S)$ = volume(S)

$L(f, P) = \sum_S m_S(f) v(S)$

lower sum

$U(f, P) = \sum_S M_S(f) v(S)$

upper sum

Easy: $L(f, P) \leq U(f, P)$ ✓

Lemma: Suppose $P' \subset P$ refinement, i.e.

each subrectangle of P' is contained in a subrectangle of P.

Then $L(f,P) \le L(f,P')$;

$U(f,P') \le U(f,P)$

Proof: Each $S \in P$ is subdivided into

$S_1, \ldots, S_\alpha$ in P', so

$v(S) = v(S_1) + \ldots + v(S_\alpha)$. also,

$m_S(f) \le m_{S_i}(f)$ since $\{f(x) : x \in S\}$

contains $\{f(x) : x \in S_i\}$.

It follows that

$m_S(f)v(S) = m_S(f)v(S_1) + \ldots + m_S(f)v(S_\alpha)$

$\le m_{S_1}(f)v(S_1) + \ldots + m_{S_\alpha}(f)v(S_\alpha)$

Summing both sides yields $L(f,P) \le L(f,P')$

The second part is basically the same.

③

Corollary: P, P' partitions, then $L(f, P') \leq U(f, P)$.

Proof: Let P'' be a refinement of both P & P'. Then

$$L(f, P') \leq L(f, P'') \leq U(f, P'') \leq U(f, P)$$

Observation: It follows that LUB of all lower sums $\leq$ GLB of all upper sums.

Definition: $f : A \to \mathbb{R}$ is integrable if

$$\sup \{L(f, P)\} = \inf \{U(f, P)\}$$

If this equality holds, we set it equal to $\int_A f \equiv \int_A f(x^1, \ldots, x^n)\, dx^1 \ldots dx^n$

④

Theorem: A bounded function $f: A \to \mathbb{R}$ is integrable iff for every $\epsilon > 0$ there is a partition P of A $\ni$ $U(f,P) - L(f,P) < \epsilon$.

Proof: If the condition holds, then

$$\sup L(f,P) = \inf U(f,P) \quad (*)$$

$$\Leftrightarrow f \text{ integrable.}$$

If f integrable, i.e. $(*)$ holds, for every $\epsilon > 0$ $\exists$ P, P' partitions $\ni$

$U(f,P) - L(f,P') < \epsilon$. If P'' refines them both,

$$U(f,P'') - L(f,P'') \leq U(f,P) - L(f,P')$$

$$< \epsilon \checkmark$$

Measure 0 and content 0:

$A \subseteq \mathbb{R}^n$ has <u>n-dim measure 0</u> if for every $\epsilon > 0$ there is a cover $\{U_1, U_2, \dots, U_m, \dots\}$ of A by closed rectangles $\ni \sum_{i=1}^{\infty} v(U_i) < \epsilon$.

$A \subseteq \mathbb{R}^n$ has content 0 if for every $\epsilon > 0$ there is a finite cover $\{U_1, \dots, U_n\}$ of A by closed rectangles $\ni \sum_{i=1}^{n} v(U_i) < \epsilon$.

Content 0 $\Longrightarrow$ Measure 0

(6)

Theorem: If $a<b$, then $[a,b] \subset \mathbb{R}$ does not have content 0. Moreover, if $\{U_1, U_2, \ldots, U_n\}$ is a finite cover of $[a,b]$ by closed intervals, then $\sum_{i=1}^{n} v(U_i) \geq b-a$

Proof: We may assume $U_i \subset [a,b]$.
Let $a = t_0 < t_1 < \ldots < t_k = b$ be all the endpoints of all U_i's. Moreover, each $[t_i, t_{i+1}]$ lies in at least one U_i, so

$$\sum_{i=1}^{n} v(U_i) \geq \sum_{j=1}^{k} (t_j - t_{j-1}) = b-a.$$

If $a<b$, $[a,b]$ does not have measure 0:

Theorem: If A is compact & has measure 0, then A has content 0.

Proof: Let $\epsilon > 0$. Since A has measure 0, there is cover $\underbrace{\{U_1, U_2, \ldots, U_n, \ldots\}}_{\text{open rectangles}}$ of A $\ni \sum_{i=1}^{\infty} v(U_i) < \infty$. Since A is compact, finitely many U_i's cover A, so

$$\sum_{i=1}^{n} v(U_i) < \epsilon.$$

Note: The conclusion is not necessarily true if A is not compact.

Ex. $\underbrace{Q}_{\text{rational numbers}} \cap [0,1]$

integration

Lemma: A closed rectangle & $f: A \to \mathbb{R}$ is bounded $\ni$ $o(f, x) < \epsilon$ $\forall x \in A$. Then $\exists$ P = partition of A w/ $U(f,P) - L(f,P) < \epsilon\, v(A)$.

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Proof: For each $x \in A$ $\exists$ U_x closed rectangle $\ni$ $x \in U_x^{o}$ & $M_{U_x}(f) - m_{U_x}(f) < \epsilon$.

Since A is compact, $U_{x_1}, \dots U_{x_n}$ cover A

Let P be a partition $\ni$ each subrectangle S of P is in some U_{x_i}. Then

$$M_S(f) - m_S(f) < \epsilon \quad \forall S, \text{ so}$$

$$U(f,P) - L(f,P) = \sum_S \left(M_S(f) - m_S(f)\right) v(S) < \epsilon\, v(A) \checkmark$$

Theorem: A closed rectangle and $f: A \to \mathbb{R}$ a bounded function. Let $B = \{x : f \text{ is not continuous at } x\}$

The f is integrable iff B is a set of measure 0.

⑨

Proof: Suppose first that B has measure 0.
Let $\epsilon > 0$ and let $B_\epsilon = \{x : o(f, x) \geq \epsilon\}$.
Then $B_\epsilon \subset B$, so B_ϵ has measure 0.
Since B_ϵ is compact (why?), B_ϵ has content 0. Thus $\exists\ U_1, U_2, \ldots, U_n$ closed rectangles such that $\bigcup U_i^{\circ}$ cover B_ϵ w/

$$\sum_{i=1}^{n} v(U_i) < \epsilon.$$

Let P be a partition of $A \ni$ every rectangle belongs to one of two categories:

i) $\mathcal{S}_1$: $S : S \subset U_i$ for some i

ii) $\mathcal{S}_2$: $S : S \cap B_\epsilon = \emptyset$

Let $|f(x)| < M$, $x \in A$. Then

$$M_S(f) - m_S(f) < 2M \text{ for every } S.$$

(10)

It follows that

$$\sum_{S\in\mathcal{J}_1} [M_S(f)-m_S(f)]\,v(S)$$

$$< 2M \sum_{i=1}^{n} v(U_i) < 2M\epsilon$$

If $S\in\mathcal{J}_2$, $o(f,x)<\epsilon \quad \forall x\in S.$

By Lemma 3.7, $\exists\, P'$, a refinement of P

$$\ni \sum_{S'\subset S} [M_{S'}(f)-m_{S'}(f)]\,v(S') < \epsilon\, v(S),$$

$$S\in\mathcal{J}_2$$

Then $U(f,P')-L(f,P') =$

$$\sum_{S'\subset S\in\mathcal{J}_1} [M_{S'}(f)-m_{S'}(f)]\,v(S')$$

$$+ \sum_{S'\subset S\in\mathcal{J}_2} [M_{S'}(f)-m_{S'}(f)]\,v(S')$$

(11)

$$< 2M\epsilon + \epsilon \sum_{S \in \mathcal{J}_2} \epsilon \cdot v(S)$$

$$\leq 2M\epsilon + \epsilon\, v(A).$$

this completes the proof.

Conversely, suppose that f is integrable. Since $B = B_1 \cup B_{\frac{1}{2}} \cup B_{\frac{1}{2}} \cup \ldots$, it is enough to show that $B_{\frac{1}{n}}$ has measure 0.

If $\epsilon > 0$, let P be a partition of A $\ni U(f,P) - L(f,P) < \frac{\epsilon}{n}$. Let $\mathcal{J}$ = collection of subrectangles S of P that intersect $B_{\frac{1}{n}}$. Then $\mathcal{J}$ covers $B_{\frac{1}{n}}$.

By constuction, if $S \in \mathcal{J}$,

$$M_S(f) - m_S(f) \geq \frac{1}{n}, \text{ so}$$

$$\frac{1}{n} \sum_{S \in \mathcal{J}} v(S) \leq \sum_{S \in \mathcal{J}} \left[M_S(f) - m_S(f)\right] v(S)$$

$$\leq \sum_{S} \left[M_S(f) - m_S(f)\right] v(S) < \frac{\epsilon}{n}$$

$$\Longleftrightarrow \sum_{S \in \mathcal{J}} v(S) < \epsilon,$$

Integrating over rectangles can be generalized w/ a little trick

If $C \subseteq \mathbb{R}^n$,

$$\chi_C(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin C \end{cases}$$

Now define $\int_C f = \int_A f \cdot \chi_C$

as long as $C \subset \underbrace{A}_{\text{rectangle}}$

Theorem: $\chi_C : A \to \mathbb{R}$ is integrable iff $\underbrace{\partial C}_{\text{boundary of } C}$ has measure 0 (and hence content 0).

Proof: If $x \in C^\circ$, then $\exists\, \underbrace{U}_{\text{open rectangle}}$ w/ $x \in U \subset C$. Thus $\chi_C = 1$ on U and χ_C is continuous at x. If $x \in$ exterior of C, $\exists\, \underbrace{U \text{ open}}_{\text{rectangle}}$ w/ $x \in U \subset \mathbb{R}^n \setminus C$.

Finally, if $x \in \underbrace{\partial C}_{\text{boundary of } C}$

then for every open rectangle U containing x, $\exists\, y_1 \in U \cap C$, so that $\chi_C(y_1) = 1$, and $\exists\, y_2 \in U \cap (\mathbb{R}^n \setminus C) \ni \chi_C(y_2) = 0$.

Hence χ_C is not continuous at x, i.e.

$\{x : \chi_C \text{ not continuous at } x\}$
$= \partial C$, so we are done by Theorem 3.8.

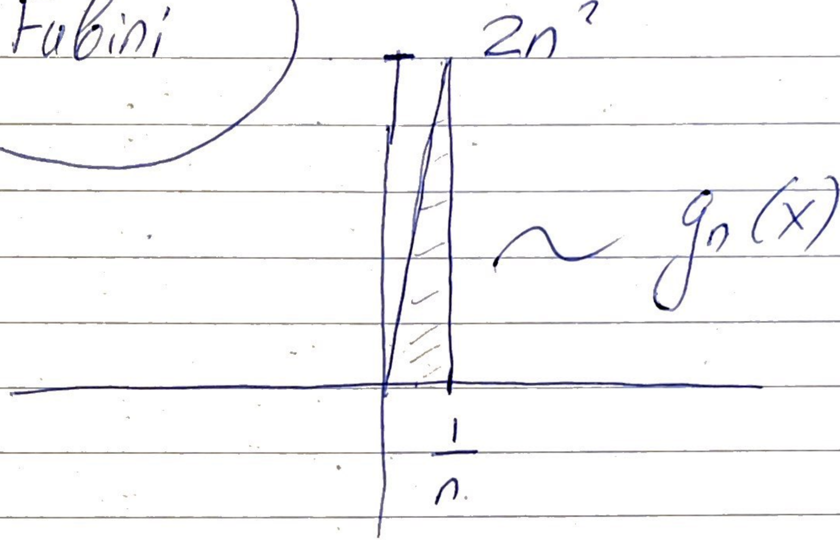
A bounded set C whose boundary has measure 0 is called Jordan measurable.

$\int_C 1$ is called the n-dim content of C.

Problem 3-11 $\longleftrightarrow$ open sets may not be Jordan measurable – a problem!

(15)

Fubini



$$\int g_n(x)\,dx = n$$

$$\lim_{n\to\infty} g_n(x) = 0$$

Let $f_1(x) = g_1(x)$, $f_n(x) = g_n(x) - g_{n-1}(x)$ $n > 1$

$$\sum_{n=1}^{N} f_n(x) = g_N(x),$$ so

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} g_N(x) = 0,$$ i.e

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)\,dx = 0.$$

On the other hand,

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\,dx = \sum_{n=1}^{\infty}\left(\int_{\mathbb{R}} (g_n(x) - g_{n-1}(x))\,dx\right)$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(x)\,dx - \int_{\mathbb{R}} g_{n-1}(x)\,dx$$

$$= \sum_{n=1}^{\infty} n - (n-1) = \infty.$$

This shows that changing the order of integration / summation is a tricky business.

Notation: $L \int_A f$ = l.u.b. of all upper sums
f bounded

$U \int_A f$ = g.l.b of all lower sums
f bounded

(17)

Fubini: Let $A \subseteq \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be closed rectangles, and let $f : A \times B \to \mathbb{R}$ be integrable. For $x \in A$, let $g_x : B \to \mathbb{R}$ be defined by $g_x(y) = f(x,y)$ and let

$$\mathcal{L}(x) = L\int_B g_x = L\int_B f(x,y)\,dy$$

$$\mathcal{U}(x) = U\int_B g_x = U\int_B f(x,y)\,dy$$

Then $\mathcal{L}$ & $\mathcal{U}$ are integrable on A and

$$\int_{A\times B} f = \int_A \mathcal{L} = \int_A \left(L\int_B f(x,y)\,dy \right) dx$$

$$\int_{A\times B} f = \int_A \mathcal{U} = \int_A \left(U\int_B f(x,y)\,dy \right) dx$$

(18)

Proof: Let P_A be a partition of A and P_B be a partition of B. Together they give a partition of $A \times B$.

It follows that

$$L(f, P) = \sum_{S} m_S(f)\, v(S) =$$

$$\sum_{S_A, S_B} m_{S_A \times S_B}(f)\, v(S_A \times S_B)$$

$$= \sum_{S_A} \left(\sum_{S_B} m_{S_A \times S_B}(f)\, v(S_B) \right) v(S_A)$$

If $x \in S_A$, $m_{S_A \times S_B}(f) \leq m_{S_B}(g_x)$, so

$$\sum_{S_B} m_{S_A \times S_B}(f)\, v(S_B) \leq \sum_{S_B} m_{S_B}(g_x) \cdot v(S_B)$$

$$\leq L \int_B g_x = \alpha(x)$$

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Therefore,

$$\sum_{S_A} \underbrace{\left(\sum_{S_B} m_{S_A \times S_B}(f) v(S_B) \right)}_{\mathcal{L}(\alpha, P_A)} v(S_A) \leq$$

$$\mathcal{L}(f, P) \leq \mathcal{L}(\alpha, P_A) \leq$$
$$\mathcal{U}(\alpha, P_A) \leq \underbrace{\mathcal{U}(u, P_A) \leq \mathcal{U}(f, P)}_{\text{same as above}}$$

Since f is integrable, $\sup \mathcal{L}(f, P) =$
$\inf \mathcal{U}(f, P) =$
$$\int_{A \times B} f$$

$\hookrightarrow \alpha$ is integrable, and
$$\int_{A \times B} f = \int_A \alpha$$

The assertion for $\mathcal{U}$ follows from

$$L(f,P) \leq L(\mathcal{L}, P_A) \leq L(\mathcal{U}, P_A)$$
$$\leq U(\mathcal{U}, P_A) \leq U(f,P)$$ ✓

Theorem 3-11: $A \subseteq \mathbb{R}^n$, $\mathcal{O}$ open cover of A. Then there is a collection Φ of C^∞ functions φ defined in an open set containing A w/

i) $0 \leq \varphi(x) \leq 1$, $x \in A$

ii) For each $x \in A$ $\exists$ V open containing $x \ni$ all but finitely many $\varphi \in \Phi$ are 0 on $\bar{V}$.

iii) For each $x \in A$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

iv) For each $\varphi \in \Phi$ $\exists$ open set $\mathcal{U}$ in $\mathcal{O}$ $\ni$ $\varphi = 0$ outside of some closed set contained in $\mathcal{U}$.

C^∞ partition of unity

(21)

An open cover $\mathcal{O}$ of an open set $A \subseteq \mathbb{R}^n$ is <u>admissible</u> if each $U \in \mathcal{O}$ is contained in A. If Φ is subordinate to $\mathcal{O}$, $f: A \to \mathbb{R}$ is bounded in some open set around each point of A, and $\{x : f \text{ discont at } x\}$ has measure 0, then each $\int_A \varphi |f|$ exists.

We define f to be integrable in the extended sense if $\sum_{\varphi \in \Phi} \int_A \varphi \cdot |f|$ converges!

Theorem: If Ψ is another partition of unity subordinate to an admissible cover $\mathcal{O}'$ of A, then $\sum_{\psi \in \Psi} \int_A \psi |f|$ also converges, and

$$\sum_{\varphi} \int_A \varphi f = \sum_{\psi} \int_A \psi f$$

(22)

ii) If A & f are bounded, then f is integrable in the extended sense.

iii) If A is Jordan-measurable and f is bounded, the this definition of $\int_A f$ agrees w/ the old one.

Change of variables:

$g: [a,b] \longrightarrow \mathbb{R}$ cont. diff

$f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) \cdot g'$$

calc 1 !

Theorem: $A \subseteq \mathbb{R}^n$ open; $g : A \to \mathbb{R}^n$ 1-1, continuously diff function such that $\det g'(x) \neq 0$ for all $x \in A$.

If $f : g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f \circ g)\, |\det g'|$$

Proof:

i) Let $\mathcal{O}$ be an admissible cover for A + for each $U \in \mathcal{O}$ and any integrable f,

$$\int_{g(U)} f = \int_U (f \circ g)\, |\det g'|$$

Then the theorem holds.

ii) If the theorem holds for $f \equiv 1$, it holds in general.

iii) If the theorem holds for $g: A \to \mathbb{R}^n$ and $g: B \to \mathbb{R}^n$, $g(A) \subseteq B$, then it is true for $h \circ g: A \to \mathbb{R}^n$.

iv) The theorem is true if g is a linear transformation.

We shall now prove the result based on the rreductions above. We shall then backtrack and take care of the reductions.

The case $n=1$ is Calc 1, so we assume the result holds when the dimension is $n-1$.

For each $a \in A$ we need only find U open $\subset A$ $\ni$ theorem holds. We may assume that

$g'(a) = I$ identity by (3ii).

(25)

Define $h: A \to \mathbb{R}^n$:

$$h(x) = (g^1(x), \ldots, g^{n-1}(x), x^n)$$

Then $h'(a) = \underline{I}$ — pushing into $\mathbb{R}^n$ from $\mathbb{R}^{n-1}$

check this assertion carefully

Hence in some open U' w/ $a \in U' \subset A$,

h is 1-1 and $\det h'(x) \neq 0$.

It follows that we can find

$K: h(U') \longrightarrow \mathbb{R}^n$:

$$K(x) = (x^1, \ldots, x^{n-1}, g^n(h^{-1}(x)))$$

and $g = \underline{\underline{K \circ h}}$.

we now need to make sure K satisfies the needed properties.

Observe that

$$(g^n \circ h^{-1})'(h(a)) = (g^n)'(a) \cdot [h'(a)]^{-1}$$

$$= (g^n)'(a),$$ and it follows that

$$D_n(g^n \circ h^{-1})(h(a)) = D_n g^n(a) = \underline{1},$$

so $k'(h(a)) = \underline{I}$

directly follows

In some open set V, w/ $h(a) \in V \subset h(U')$,

the function k is 1-1

and $\det k'(x) \neq 0$.

Let $U = k^{-1}(V)$, so $g = k \circ h$,

$h : U \longrightarrow \mathbb{R}^n$, $k : V \longrightarrow \mathbb{R}^n$,

$h(U) \subset V$.

By iii), it suffices to prove the theorem for h and K. The proofs are basically the same.

<u>Proof for K:</u>

Let $W \subset U$ be a rectangle of the form $D \times [a_n, b_n]$, $D =$ rectangle in $\mathbb{R}^{n-1}$.

By Fubini,

$$\int_{h(W)} 1 = \int_{[a_n, b_n]} \left(\int_{h(D \times \{x^n\})} 1 \, dx^1 \dots dx^{n-1} \right) dx^n$$

Let $h_{x^n}(x^1, \dots, x^{n-1}) = \underbrace{(g^1(x^1, \dots, x^n), \dots, g^{n-1}(x^1, \dots, x^n))}_{\text{clearly } 1\text{-}1}$

and $\det (h_{x^n})'(x^1, \dots, x^{n-1}) = \det h'(x^1, \dots, x^n) \neq 0$

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Moreover, $\displaystyle\int_{h(D\times\{x^n\})} 1 \, dx^1 \dots dx^{n-1}$

$$= \int_{h_{x^n}(D)} 1 \, dx^1 \dots dx^{n-1}.$$

Induction:

$$\int_{h(W)} 1 = \int_{[a_n, b_n]} \left(\int_{h_{x^n}(D)} 1 \, dx^1 \dots dx^{n-1} \right) dx^n$$

$$= \int_{[a_n, b_n]} \left(\int_D |\det (h_{x^n})'(x^1, \dots, x^{n-1})| \, dx^1 \dots dx^{n-1} \right) dx^n$$

$$= \int_{[a_n, b_n]} \left(\int_D |\det h'(x^1, \dots, x^n)| \, dx^1 \dots dx^{n-1} \right) dx^n$$

$$= \int_W |\det h'| \quad \checkmark$$

We are not done because we need to prove i), ii), iii) & iv)!

i) Suppose there is an admissible cover $\mathcal{O}$ for $A \ni$ for each $U \in \mathcal{O}$ and any integrable f we have

$$\int_{g(U)} f = \int_U (f \circ g)\,|\det g'|$$

Then the theorem is true for all of A.

Proof: The collection of all $g(U)$ is an open cover of $g(A)$. Let Φ be a partition of unity subordinate to this cover. If $\varphi = 0$ outside of $g(U)$, then since g is 1-1, we have $(\varphi \cdot f) \circ g = 0$ outside of U.

$$\int_{g(U)} \varphi \cdot f = \int_U [(\varphi \cdot f) \circ g] \, |\det g'|$$

can be rewritten as

$$\int_{g(A)} \varphi \cdot f = \int_A [(\varphi \cdot f) \circ g] \, |\det g'|$$

It follows that

$$\int_{g(A)} f = \sum_{\varphi \in \Phi} \int_{g(A)} \varphi \cdot f =$$

$$\sum_{\varphi \in \Phi} \int_A [(\varphi \cdot f) \circ g] \, |\det g'|$$

$$= \sum_{\varphi \in \Phi} \int_A (\varphi \circ g)(f \circ g) |\det g'|$$

$$= \int_A f \circ g \, |\det g'|$$

(31)

ii) It suffices to prove the theorem for $f \equiv 1$.

Proof: Note that the condition in i) is equivalent to

$$\int_V f = \int_{g^{-1}(V)} (f \circ g) |\det g'|$$

for V in an admissible cover of $g(A)$.

To prove ii) note that if the theorem holds for $f = 1$, it holds for $f =$ constant

Let $V =$ rectangle in $g(A)$ and $P =$ partition of V.

For each subrectangle S of P, let

$$f_S \equiv \underbrace{m_S(f)}_{\text{constant}}$$

We get

$$L(f,P) = \sum_S m_S(f)\, v(S) =$$

$$\sum_S \int_{S^\circ} f_S = \sum_S \int_{g^{-1}(S^\circ)} (f_S \circ g)\,|\det g'|$$

$$\leq \sum_S \int_{g^{-1}(S^\circ)} (f \circ g)\,|\det g'|$$

$$\leq \int (f \circ g)\,|\det g'|.$$

This proves that $\int_V f \leq \int_{g^{-1}(V)} f \circ g \cdot |\det g'|$

The same argument w/ $f_S = M_S(f)$ shows that

$$\int_V f \geq \int_{g^{-1}(V)} (f \circ g)\,|\det g'|$$

This completes the proof.

Proof of iii)

If the theorem is true for $g: A \to \mathbb{R}^n$ & $h: B \to \mathbb{R}^2$, $g(A) \subset B$, then it is true for $h \circ g: A \to \mathbb{R}^n$.

Proof of iii)

$$\int_{h\circ g(A)} f = \int_{h(g(A))} f = \int_{g(A)} (f\circ h)\,|\det h'|$$

$$= \int_A [(f\circ h)\circ g][|\det h'|\circ g]\cdot|\det g'|$$

$$= \int_A f\circ(h\circ g)\,|\det(h\circ g)'|$$

iv) The theorem is true if g is a linear transformation.

Proof: Enough to show that

$$\int_{g(U)} 1 = \int_U |\det g'| \qquad U = \text{open rectangle}$$

This is problem 3-35 (homework)

Sard's theorem: Let $g: A \to \mathbb{R}^n$ cont. diff,

$A \subseteq \mathbb{R}^n$ open; $B = \{x \in A : \det g'(x) = 0\}$

Then $g(B)$ has measure 0.

Proof: Let $U \subset A$ closed rectangle w/ all sides equal to ℓ

i.e. a cube

Let N be sufficiently large, and divide U into N^n rectangles of side-length ℓ/N. Then for each such rectangle S, if $x \in S$,

$$|Dg(x)(y-x) - g(y) - g(x)| < \epsilon|x-y|$$

$$\leq \epsilon\sqrt{n}\,\frac{\ell}{N}$$

Definition of derivative + geometry

If S intersects B, we choose $x \in S \cap B$.

Since $\det g'(x) = 0$, $\{Dg(x)(y-x) : y \in S\}$

$\cap$

$(n-1)$-dim subspace V of $\mathbb{R}^n$.

Therefore, $\{g(y)-g(x) : y \in S\}$ lies within $\epsilon\sqrt{n}\,\frac{\ell}{N}$ of V, so

$\{g(y) : y \in S\}$ lies within $\epsilon\sqrt{n}\,\frac{\ell}{N}$ of the $(n-1)$-plane $V + g(x)$.

By Lemma 2-10, some M

$|g(x)-g(y)| < \tilde{M}|x-y| \leq M\sqrt{n}\,\frac{\ell}{N}$

It follows that if S intersects B,

$\{g(y) : y \in S\} \subset$ cylinder w/ height $< 2\epsilon\sqrt{n}\left(\frac{\ell}{N}\right)$ and base

given by an $(n-1)$-dim sphere of radius $< M\sqrt{n}\left(\frac{\ell}{N}\right)$.

This cylinder has volume $< C\left(\frac{\ell}{N}\right)^n \epsilon \cdot N^n$
$= C\ell^n \cdot \epsilon$.

Since this is true for all ϵ,

$g(U \cap B)$ has measure 0, so the conclusion follows by Theorem 3-4 (countable union of sets of measure 0 has measure 0).