COVERING THE PLANE BY ROTATIONS OF A LATTICE ARRANGEMENT OF DISKS

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Abstract. Suppose we put an ϵ-disk around each lattice point in the plane, and then we rotate this object around the origin for a set Θ of angles. When do we cover the whole plane, except for a neighborhood of the origin? This is the problem we study in this paper. It is very easy to see that if $\Theta = [0, 2\pi]$ then we do indeed cover. The problem becomes more interesting if we try to achieve covering with a small closed set Θ.

1. Introduction

In this paper we discuss problems of covering the plane, or all but a bounded part of it, by rotations of fattened lattices.

Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice in the plane (a discrete subgroup of $\mathbb{R}^2$, of dimension 2) and $\epsilon > 0$ be a small number. We define the fattened lattice

$$E = E(\Lambda, \epsilon) = \Lambda + B_\epsilon(0),$$

as the ϵ neighborhood of Λ (here $B_\epsilon(0) = \{x \in \mathbb{R}^2 : |x| < \epsilon\}$).

Suppose, as we shall do throughout this paper, that Θ is a set of angles, viewed as a subset of S^1, the unit circle in the plane. We shall always assume that Θ is a closed set (see the remark after Definition 1). If R_θ denotes the rotation by θ and

$$R_\Theta E = \{R_\theta x : \theta \in \Theta, x \in E\},$$

the question we are interested in is when $R_\Theta E$ contains the complement of a disk, when, in other words, E rotated by the angles in Θ covers everything except the only obvious obstacle, a neighborhood of the origin.

It is easy to see, and left to the reader, that if we rotate by all possible angles, namely if we take $\Theta = S^1$, then we do indeed achieve covering. The question becomes interesting if we try to achieve the same with a *small* closed set Θ.

This problem was motivated by earlier results on distances appearing between points of a set of positive upper density. In fact, a question raised by Sz. Révész was whether for any set E of positive upper density, the union of finitely many rotates of $E - E$ can cover the complement of a disk. We answer this question in the negative (Theorem 2). The first positive result we obtained in this circle of problems (Corollary 1), was deduced easily using a result (Theorem 1) which speaks about which distances are realizable in sets of positive upper density in Euclidean spaces. Theorem 1 was obtained in [4] by a careful rewriting of an earlier result of Bourgain [1] who had improved on Falconer and Marstrand [2] and Furstenberg, Katznelson and Weiss [3].

Definition 1. The set of angles $\Theta \subseteq S^1$ is called (Λ, ϵ)-*good* if $R_\Theta E$ contains the complement of a disk, where $E = \Lambda + B_\epsilon(0)$. The set Θ will be called *good* if it is (Λ, ϵ)-good for all lattices Λ and $\epsilon > 0$.

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It is easy to see that Θ is (Λ, ϵ)-*good* if and only if its closure $\overline{\Theta}$ is (Λ, ϵ)-*good*. Therefore, we restrict our attention to closed sets throughout this paper.

In summary our results are as follows.

(1) If $\Theta \subseteq S^1$ is any arc then Θ is good (Corollary 1). This follows from using Theorem 1 which was proved in [4]. We also give an elementary proof of Corollary 1 in §3.
(2) Using Corollary 1 we prove in Corollary 3 that for any Λ there are sets $\Theta \subseteq S^1$, which consist of a convergent sequence of angles plus its limit point, and which are (Λ, ϵ)-good for all positive ϵ.
(3) If $\Theta \subseteq S^1$ is finite then Θ is not (Λ, ϵ)-good for any lattice Λ and any ϵ smaller than half the shortest non-zero vector in Λ (Theorem 2).
(4) For any lattice Λ and any ϵ which is smaller than half the shortest non-zero vector in Λ there exists an infinite closed set $\Theta \subseteq S^1$ which is not (Λ, ϵ)-good (Corollary 4).
(5) If $\Theta \subseteq S^1$ is rich enough to support a probability measure whose Fourier Transform is small near infinity (depending on Λ and ϵ) then Θ is (Λ, ϵ)-good (Theorem 4). Since any arc of S^1 supports probability measures whose Fourier Transform tends to 0 this is a new proof of Corollary 1. Theorem 4 is proved directly and not by appealing to any results on distance sets.
(6) If $\Theta \subseteq S^1$ has positive one-dimensional measure then it is good (Corollary 5).
(7) There are sets $\Theta \subseteq S^1$ of 0 one-dimensional measure (even sets of 0 Hausdorff dimension) which are good (Corollary 6).

Open problem: Let $\epsilon > 0$ and $E = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{R}\} + B_\epsilon(0)$. Is there a finite set of angles $\theta_1, \ldots, \theta_n$ such that

$$\bigcup_{j=1}^{n} R_{\theta_j} E$$

covers the plane?

One might try to prove that this is not the case by showing that in any such finite set of rotations of E *any* line $y = \alpha x$ which is not parallel to any of the strips cannot be covered. This amounts to covering the real line by finitely many dilates of the function $f(x) = \sum_{n \in \mathbb{Z}} \chi_{(-\epsilon, \epsilon)}(x - n)$. This is indeed possible, for any $\epsilon > 0$, so this approach to the open problem above fails.

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2. Continuous moving

The purpose of this section is to show that any arc is good. A probability measure is called δ-good if its Fourier Transform is $< \delta$ near infinity. In [4] the following theorem is proved (but not stated in this form).

Theorem 1. *Suppose that $E \subseteq \mathbb{R}^d$, $d \geq 2$, has upper density equal to $\epsilon > 0$ and that the 0-symmetric convex body K affords a $(C_d \epsilon)$-good probability measure σ supported on its boundary (the constant C_d depends on the dimension only). Then, there exists a nonnegative number t_0 such that for all $t \geq t_0$ there exist $x, y \in E$ with*

$$\|x - y\|_K = t \quad \textit{and} \quad \frac{x - y}{\|x - y\|_K} \in \mathrm{supp}\sigma.$$

Corollary 1. *Suppose that $\Lambda = A\mathbb{Z}^2 \subseteq \mathbb{R}^2$ is a lattice and $\epsilon > 0$. Write $E = \Lambda + B_\epsilon(0)$. Then, for any arc $\Theta \subseteq S^1$ we have $B_{t_0}^c \subseteq R_\Theta E$, for some $t_0 > 0$.*

Proof. Assume $\Theta = [-\theta_0, \theta_0]$. Let $\Gamma = [a, b]$ be an arc of S^1 of length smaller than θ_0 and take a smooth probability measure σ on S^1 whose support is $[a, b]$. Since $\widehat{\sigma}$ tends to 0 at

∞ we can apply Theorem 1 to the set $E' = \Lambda + B_{\epsilon/2}(0)$ and σ and we get that there is t_0 such that for any $t \geq t_0$ we have (notice that $E = E' - E'$)

$$t\Gamma \cap E \neq \emptyset.$$

This implies that

$$t\Gamma \subseteq R_{[-\theta_0,\theta_0]}E, \quad \text{for } t \geq t_0.$$

Since finitely many rotations of Γ will cover S^1, it follows by applying our Theorem 1 finitely many times and taking the maximum t_0 that there is a finite t_0' such that any vector of length $\geq t_0'$ is in $R_{[-\theta_0,\theta_0]}E$. $\square$

3. Elementary proof of Corollary 1

We will give the elementary proof for the lattice $\Lambda = \mathbb{Z}^2$ for simplicity. The same idea applies to any other lattice, too.

The covering

$$B_{t_0}^c \subseteq R_{[-\theta_0,\theta_0]}E$$

is clearly equivalent to the fact that each 'annulus-arc' $A_{t,\gamma} = \{(r,\phi) : \ t < r < t+\epsilon, \gamma \leq \phi \leq \gamma + 2\theta_0\}$ (given in polar coordinates) contains a lattice point for any $t > t_0$ and any γ.

Take finitely many points $n_j = (\cos\alpha_j, \sin\alpha_j)$, $j = 1, \ldots N$ on the unit circle such that $\tan\alpha_j$ is irrational, and every open arc of length θ_0 contains at least one n_j. Consider the lines $y = -\frac{1}{\tan\alpha_j}x$ on the torus $\mathbb{T} = [0,1] \times [0,1]$. Each of these lines form a dense set on the torus, therefore there exist numbers $h_j > 0$ such that the line-segments $S_1 = \{(x,y) : \ y = -\frac{1}{\tan\alpha_j}x; \ x \in [0,h_j]\}$ and also $S_2 = \{(x,y) : \ y = -\frac{1}{\tan\alpha_j}x; \ x \in [-h_j, 0]\}$ are already $\epsilon/4$ dense in $\mathbb{T}$ (i.e. for every $q \in \mathbb{T}$ there is a point s of the segment such that $|s - q| < \epsilon/4$; equivalently, the $\epsilon/4$-neighbourhood of S_1, S_2 already covers the whole torus). Let $H = \max\{h_j : \ j = 1, \ldots N\}$. It follows, by construction, that for each j the $\epsilon/4$-neighbourhood of *any line segment* (i.e. not necessarily starting from the origin) of length H and steepness $-\frac{1}{\tan\alpha_j}$ covers the whole torus.

Take now any $A_{t,\gamma}$. There is an α_j such that $\gamma < \alpha_j < \gamma + 2\theta_0$. Consider the point p with polar coordinates $p = (t + \epsilon/2, \alpha_j) \in A_{t,\gamma}$. It is clear from plane geometry that if t is large enough then there there is a strip S of steepness $-\frac{1}{\tan\alpha_j}$ and half-width $\epsilon/4$ and length H, starting from p (in one of the directions along the line with steepness $-\frac{1}{\tan\alpha_j}$), which remains fully inside $A_{t,\gamma}$. By construction, this strip covers the whole torus, and hence contains a lattice point.

4. Covering using a convergent sequence of rotation angles

The following is a consequence of Corollary 1 which was shown to us by Y. Katznelson.

Corollary 2. *Suppose that $\Lambda \subseteq \mathbb{R}^2$ is a lattice, $\epsilon > 0$ Let I be any arc in S^1. We can find a convergent sequence of angles $\theta_n \in I$, $n = 1, 2, \ldots$, such that the set $\Theta = \overline{\{\theta_n, n = 1, 2, \ldots\}}$ is (Λ, ϵ)-good.*

Proof. Write $E = \Lambda + B_\epsilon(0)$. Choose any sequence of arcs $I_n \subseteq I$ which converge to a single point $\theta' \in I$. From Corollary 1 there is an increasing sequence of numbers $r_n \to \infty$ such that

$$B_{r_n}^c \subseteq R_{I_n}E.$$

Let F_n be a finite subset (by compactness such a subset exists) of I_n such that

$$\overline{B_{r_{n+1}}} \setminus B_{r_n} \subseteq R_{F_n}E.$$

It follows that the countable set $F = \bigcup_{n=1}^\infty F_n$ is such that

$$B_{r_1}^c \subseteq R_F E.$$

Obviously F is a sequence that converges to θ'. $\square$

Corollary 2 can be strengthened as follows.

Corollary 3. *For any lattice $\Lambda \subseteq \mathbb{R}^2$ we can find a convergent sequence of angles θ_n such that the set $\Theta = \overline{\{\theta_n, n = 1, 2, \ldots\}}$ is (Λ, ϵ)-good for all $\epsilon > 0$.*

Proof. Pick a positive sequence $a_n \to 0$ and, using Corollary 2, find a set $\Theta_n \subseteq (0, a_n)$, which consists of a sequence convergent to 0, such that Θ_n is $(\Lambda, 1/n)$-good. Clearly the set $\bigcup_{n=1}^{\infty} \Theta_n$ is a sequence which converges to 0 and is (Λ, ϵ)-good for all positive ϵ. $\square$

5. Finitely many rotations are never enough, nor are some infinite sets

Theorem 2. *Let Λ be a lattice in $\mathbb{R}^d$, $d \geq 2$, and $\epsilon > 0$ be smaller than $s(\Lambda)/2$, where $s(\Lambda)$ is the length of the shortest non-zero vector of Λ. Write as usual $E = \Lambda + B_\epsilon(0)$. Then it is impossible to find a finite set of orthogonal matrices $O_1, \ldots, O_n$ such that $\bigcup_{j=1}^n O_j E$ contains the complement of a ball.*

Proof. Suppose $B_r(0)^c \subseteq \bigcup_{j=1}^n O_j E$.

Let $\epsilon < \epsilon' < s(\Lambda)/2$ and take $\phi \geq 0$ to be a continuous function with supp$\phi = B_{\epsilon'}(0)$ which is ≥ 1 on $B_\epsilon(0)$. Then the functions $f_j(x) = \sum_{\lambda \in O_j \Lambda} \phi(x - \lambda)$ are periodic continuous functions and writing $f = \sum_{j=1}^n f_j$ we have

$$B_r(0)^c \subseteq \bigcup_{j=1}^n O_j E \subseteq \left\{ x \in \mathbb{R}^d : f(x) \geq 1 \right\}. \tag{1}$$

It follows that f is almost-periodic hence there are arbitrarily large vectors $T \in \mathbb{R}^d$ such that $\|f(x) - f(x - T)\|_{L^\infty(\mathbb{R}^d)} \leq 1/2$. But there is an annular neighborhood of 0 where $f = 0$. By the almost periodicity of f this implies that there are translates of this neighborhood arbitrarily far where $f \leq 1/2$, and this contradicts (1). $\square$

Using Theorem 2 we can prove the following.

Corollary 4. *Assume the notations of Theorem 2 and let Λ and ϵ be fixed, with $\epsilon < s(\Lambda)/2$. Then there is an infinite $\Theta \subseteq S^1$ such that the set $R_\Theta E$ is not (Λ, ϵ)-good.*

Proof. We only sketch the proof. Our set Θ will be $\overline{\{\theta_1, \theta_2, \ldots\}}$, where θ_n is a convergent sequence. Let θ_1 be arbitrary and assume that we have already chosen the angles $\theta_1, \ldots, \theta_n$. By Theorem 2 we know that there are "holes" arbitrarily far from the origin, i.e. open regions of the plane which are not in $R_{\{\theta_1, \ldots, \theta_n\}} E$. Choose θ_{n+1} distinct from $\theta_1, \ldots, \theta_n$ but so close to, say, θ_n that the set $R_{\{\theta_1, \ldots, \theta_n, \theta_{n+1}\}} E$ barely touches all holes up to distance n from the origin. For instance, we may arrange that the inradius r of each hole stays at least $(1 - 10^{-n})r$. This construction implies the preservation of all the holes in $R_\Theta E$. $\square$

There are even uncountable sets which are not good for covering.

Theorem 3. *Assume the notations of Theorem 2 and let Λ and ϵ be fixed, with $\epsilon < s(\Lambda)/2$. Then there is a perfect set $\Theta \subseteq S^1$ which is not (Λ, ϵ)-good.*

Proof. We will construct a Cantor-type set $\Theta = \cap_{j=1}^{\infty} \Theta_j$ which is not (Λ, ϵ)-good. Let $I_1 = [\psi_1, \psi_1 + \alpha_1]$ be any closed arc in S^1 of length $|I_1| = \alpha_1 > 0$, and let $\Theta_1 = I_1$. Take $R_1 > 0$ sufficiently large, and consider any point $z_1 \in \mathbb{R}^2$ with polar coordinates $z_1 = (R_1, \phi_1)$, i.e. $|z_1| = R_1$. Consider the arc $A_1 = \{(r, \phi) : \ r = R_1, \phi_1 - \psi_1 - \alpha_1 \leq \phi \leq \phi_1 - \psi_1$. Clearly, this arc cannot fully be contained in $E = \Lambda + B_\epsilon(0)$. Take two closed disjoint sub-arcs $S_1 = R_1 \cdot [\beta_1, \gamma_1]$, $S_2 = R_1 \cdot [\beta_2, \gamma_2]$ of A_1 which are disjoint from E. Let $I_{2,1} = [\phi_1 - \gamma_1, \phi_1 - \beta_1]$ and $I_{2,2} = [\phi_1 - \gamma_2, \phi_1 - \beta_2]$ be the closed intervals of angles in Θ_1 corresponding to the subarcs S_1 and S_2. Let $\Theta_2 = I_{2,1} \cup I_{2,2}$. Then, by construction, $\Theta_2 \subset \Theta_1$ and $R_{\Theta_2} E$ will not contain z_1.

We continue this procedure inductively. Assume that Θ_n is already given as the disjoint union of 2^{n-1} closed intervals, such that $\Theta_1 \supset \Theta_2 \cdots \supset \Theta_n$ and there are points z_j such that $|z_j| > 2^{j-1}R_1$ and $z_j \notin R_{\Theta_{j+1}}E$. Then take $R_n > 2^n R_1$ sufficiently large, and consider any point $z_n \in \mathbb{R}^2$ with polar coordinates $z_n = (R_n, \phi_n)$, i.e. $|z_n| = R_n$. The set $A_n = R_n \cdot [\phi_n - \Theta_n]$ consists of 2^n arcs corresponding to the intervals of Θ_n. If R_n is large enough then none of these arcs can fully be contained in E. We can therefore choose two closed subarcs in each of them which are disjoint from E. Then Θ_{n+1} will be defined as the union of 2^{n+1} intervals of angles corresponding to these subarcs, as above.

Define $\Theta = \cap_{j=1}^{\infty}\Theta_j$. Then Θ is a closed set of continuum many points and, by construction, $z_n \notin R_\theta E$ for all n. ☐

6. Covering when carrying "good" measures

Theorem 4. *Assume Λ is a lattice in the plane and $\epsilon > 0$. Write $E = \Lambda + B_\epsilon(0)$. Then there is $0 < \delta(\epsilon) \sim \operatorname{dens}\Lambda \cdot \epsilon^2$ (as $\epsilon \to 0$) such that if $\Theta \subseteq S^1$ carries a probability measure σ with $\limsup_{\xi\to\infty} |\widehat{\sigma}(\xi)| < \delta(\epsilon)$ then the set $R_\Theta E$ contains the complement of a disk.*

Proof. Let $\phi \geq 0$ be a C^∞ function supported in $B_1(0)$ satisfying $\phi(0) = \widehat{\phi}(0) = 1$ and with $\widehat{\phi} \geq 0$. Write $\phi_r(x) = r^{-2}\phi(x/r)$ which also has integral 1 and is supported in $B_r(0)$. For large $q > 0$ define

$$f(x) = f_q(x) = \phi_q \cdot (\phi_\epsilon * \delta_\Lambda), \quad \text{where } \delta_\Lambda = \textstyle\sum_{\lambda\in\Lambda} \delta_\lambda.$$

It is sufficient to show that if $|x|$ is sufficiently large then there is $q > 0$ such that

$$\int f(R_\theta x)\, d\sigma(\theta) > 0,$$

or, equivalently, that

$$\int f\left(|x| R_{x/|x|}\theta\right)\, d\sigma(\theta) > 0. \tag{2}$$

Evaluating (2) on the Fourier side and applying a change of variable we can rewrite (2) as

$$\int \widehat{f}(\xi)\widehat{\sigma}(|x|R_{x/|x|}\xi)\, d\xi > 0. \tag{3}$$

From the definition of f and the Poisson summation formula

$$\widehat{\delta_\Lambda} = \operatorname{dens}\Lambda \cdot \delta_{\Lambda^*}$$

(where $\Lambda^* = A^{-\top}\mathbb{Z}^2$ is the dual lattice) we get $\widehat{f} = \operatorname{dens}\Lambda \cdot \widehat{\phi_q} * (\widehat{\phi_\epsilon} \cdot \delta_{\Lambda^*})$.

Thus the left hand side of (3), apart from a factor $\operatorname{dens}\Lambda$, can be written as

$$\sum_{\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda) \int \widehat{\phi}(q(\xi-\lambda))\widehat{\sigma}(|x|R_{x/|x|}\xi)\, d\xi = \overbrace{(\text{term for } \lambda = 0)}^{I} + \overbrace{\sum_{0\neq\lambda\in\Lambda^*} \cdots}^{II} \tag{4}$$

Since $q^2\widehat{\phi}(q\xi)$ is an approximate identity, with x fixed and $q \to \infty$ we have

$$I = \int \widehat{\phi}(q\xi)\widehat{\sigma}(|x|R_{x/|x|}\xi)\, d\xi \sim q^{-2}.$$

This will be the main term in the right hand side of (4).

Write $m(r) = \sup_{|z|\ge r} |\widehat{\sigma}(z)|$. Our assumption is that $\limsup_{r\to\infty} m(r) \le \delta(\epsilon)$. For II we have

$$\begin{aligned} II &= \int \widehat{\sigma}(|x| R_{x/|x|}\xi) \sum_{0\ne\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda)\widehat{\phi}(q(\xi-\lambda))\,d\xi \\ &\le \int m(|x||\xi|) \sum_{0\ne\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda)\widehat{\phi}(q(\xi-\lambda))\,d\xi \end{aligned}$$

Write $G(\xi) = \sum_{0\ne\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda)\widehat{\phi}(q(\xi-\lambda))$ and let r_0 be the length of the shortest non-zero vector in Λ^* and $B = B_{r_0/2}(0)$. Then

$$\begin{aligned} II &\le \int m(|x||\xi|) G(\xi)\,d\xi \\ &\le \int_B G(\xi)\,d\xi + \int_{B^c} m(|x||\xi|)G(\xi)\,d\xi \\ &= I_1 + I_2. \end{aligned}$$

To estimate I_1 we use the fact that $\widehat{\phi} \ge 0$ and the balls $\lambda + B$ are disjoint, $\lambda \in \Lambda^*$:

$$\begin{aligned} I_1 &= \sum_{0\ne\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda) \int_B \widehat{\phi}(q(\xi-\lambda))\,d\xi \\ &\le \int_{B+(\Lambda^*\setminus\{0\})} \widehat{\phi}(q\xi)\,d\xi \\ &= q^{-2} \int_{q(B+(\Lambda^*\setminus\{0\}))} \widehat{\phi}(\eta)\,d\eta \\ &\le q^{-2} \int_{(qB)^c} \widehat{\phi}(\eta)\,d\eta \\ &\le o(q^{-2}) \quad \text{(by the rapid decay of } \widehat{\phi}). \end{aligned}$$

Finally, for I_2 we use our assumption about $m(\cdot)$ and the estimate

$$\begin{aligned} \int G(\xi)\,d\xi &\le \sum_{\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda) \int \widehat{\phi}(q(\xi-\lambda))\,d\xi \\ &= q^{-2} \sum_{\lambda\in\Lambda^*} \widehat{\phi}(\epsilon\lambda) \\ &= C(\epsilon)q^{-2}, \end{aligned}$$

where $C(\epsilon) \sim \mathrm{vol}\,\Lambda \cdot \epsilon^{-2}$ as $\epsilon \to 0$. This shows that $I_2 \le C(\epsilon)q^{-2}m(|x|r_0/2)$ and, if $m(r)$ is smaller than $\delta(\epsilon) := 1/C(\epsilon)$ near infinity, then there is a value $R > 0$ such that $|x| > R$ implies that (3) holds for some large q, as I will be the dominant term in (4). □

Corollary 5. *Suppose $\Theta \subseteq S^1$ is a closed set with positive one-dimensional measure. Then Θ is good.*

Proof. By Theorem 4 it is enough to construct, for any $\delta > 0$, a probability measure μ_δ supported on Θ whose FT is at most δ in a neighborhood of ∞.

For this let x be a Lebesgue point of Θ and let the x-centered arc $J \subseteq S^1$ be such that Θ has density $> 1 - \frac{\delta}{10}$ in J. Let ϕ be a nonnegative smooth function supported on J such that the L^1 distance of ϕ and χ_J is bounded by $(\delta|J|)/10$ and $\int \phi = |J|$.

Define the following probability measures:

$$\mu = \frac{\chi_J}{|J|},\ \nu = \frac{\phi}{|J|},\ \mu_\delta = \frac{\chi_{\Theta\cap J}}{|\Theta \cap J|}.$$

By our choice of J and ϕ it is clear that

$$\|\mu - \mu_\delta\| < \frac{\delta}{2},\ \|\mu - \nu\| < \frac{\delta}{2},$$

hence we also have $\|\nu - \mu_\delta\| < \delta$. Since $\widehat{\nu}(\xi) \to 0$ as $\xi \to \infty$ it follows that μ_δ has FT which is at most δ in a neighborhood of ∞, as required. □

7. Existence of good sets of rotations of measure 0

We owe the following result to Y. Katznelson.

Theorem 5. *For any arc in S^1 there exists a set Θ of one-dimensional measure 0 contained in that arc which carries a probability measure σ whose Fourier Transform tends to 0.*

Proof. We shall construct σ as a weak limit point of a sequence of probability measures μ_n whose Fourier Transform tends to 0 at ∞. We set μ_1 to be arc-length on the given interval, smoothly cut-off by a positive function and normalized to be a probability measure. It follows that $\widehat{\mu_1}(\xi) \to 0$ as $|\xi| \to \infty$.

Suppose we have constructed the measure μ_n and its support is the union of arcs $I_1^{(n)}, I_2^{(n)}, \ldots, I_{m_n}^{(n)}$. Assume $\left|I_1^{(n)}\right| \ge \left|I_2^{(n)}\right| \ge \cdots \ge \left|I_{m_n}^{(n)}\right|$.

The next measure μ_{n+1} will be equal to μ_n on the arcs $I_2^{(n)}, \ldots, I_{m_n}^{(n)}$. In $I_1^{(n)}$ the measure μ_n will be replaced by a measure which will be supported by a union of sub-arcs of $I_1^{(n)}$, all of them shorter than $I_{m_n}^{(n)}$.

Let $R_n \ge \max\{n, R_{n-1}\}$ be such that $|\widehat{\mu_n}(\xi)| \le 1/n$ for all ξ with $|\xi| \ge R_n$. To get μ_{n+1} from μ_n in the arc $I_1^{(n)}$ we subdivide $I_1^{(n)}$ into $N \ge 2$ equal intervals and in each of them, say in $[a,b]$, we shift all the mass of μ_n into a smooth positive bump in the interval $[a,c]$, where $c - a = \min\left\{(b-a)/2, \left|I_{m_n}^{(n)}\right|/2\right\}$. Clearly we can choose N so large that

$$\left|\widehat{\mu_{n+1}}(\xi) - \widehat{\mu_n}(\xi)\right| \le 2^{-n}/n, \quad (|\xi| \le R_n). \tag{5}$$

The reason is that if N is large enough the functions $e_\xi(x) = e^{2\pi i\langle \xi, x\rangle}$, $|\xi| \le R_n$, are almost constant for x in the arc $[a,b]$.

The new measure μ_{n+1} is supported on the finitely many intervals

$$I_1^{(n+1)} = I_2^{(n)}, \ldots, I_{m_n-1}^{(n+1)} = I_{m_n}^{(n)}$$

followed by the new intervals $I_{m_n}^{(n+1)}, \ldots, I_{m_{n+1}}^{(n+1)}$ that came from $I_1^{(n)}$. Its Fourier Transform stil tends to 0 at ∞.

It is clear from the construction that $\mu_n(I_1^{(n)}) \to 0$.

Suppose now that $R_n \le |\xi| < R_{n+1}$. By the definition of R_n we have

$$|\widehat{\mu_n}(\xi)| \le 1/n. \tag{6}$$

Since the measures μ_n and μ_{n+1} only differ in $I_1(n)$ we have

$$\left|\widehat{\mu_{n+1}}(\xi) - \widehat{\mu_n}(\xi)\right| \le \mu_n(I_1^{(n)}) = \mu_{n+1}(I_1^{(n)}). \tag{7}$$

Finally, if $k \ge 1$, applying (5) repeatedly we obtain

$$\left|\widehat{\mu_{n+1+k}}(\xi) - \widehat{\mu_{n+1}}(\xi)\right| \le 2/n. \tag{8}$$

Combining (6), (7) and (8) we obtain

$$|\widehat{\mu_k}(\xi)| \le \epsilon_n := 2/n + \mu_n(I_1^{(n)}), \quad (k \ge n). \tag{9}$$

Suppose now that σ is a weak limit of a subsequence of μ_n. We have shown that if $R_n \le \xi < R_{n+1}$ then $\widehat{\sigma}(\xi) \le \epsilon_n$. Since $R_n \to \infty$ and $\epsilon_n \to 0$ we have proved that the Fourier Transform of σ tends to 0 at infinity. Finally, the support of σ is contained in the support of μ_n for infinitely many n, and hence it has Lebesgue measure 0. □

The following is now immediate from Theorem 4 combined with Theorem 5.

Corollary 6. *In any arc of S^1 one can find a good set Θ of one-dimensional measure 0.*

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