The Box Problem in Two and Higher Dimensions

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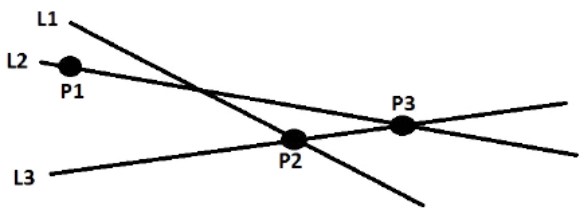
1 Introduction

In $\mathbb{F}^2$, two lines intersect at no more than one point. This means that two lines can share at most one incidence between a point and the lines. Given a set of n lines and n points, this fact can be used to discover the upper bound on the number of possible incidences between lines and points. We can understand this case of n lines and n points by considering a matrix of ones and zeros. Let us call our matrix I with I_{ij} indicating the value in the i^{th} row and j^{th} column. We will designate that the points are represented by rows and the lines by the columns (although the reverse would also be valid). We represent an incidence of the point i and the line j by setting $I_{ij} = 1$. For example, the arrangement of lines



can be represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The condition of having no more than one intersection between any two lines is the same as saying we cannot have a "rectangle" of ones, ie that $I_{ij}I_{ij'}$ = 1 for at most one value of i. Equivalently, this condition says that $I_{ij}I_{ij'}I_{i'j}I_{i'j'} = 0$ for all values of i, i', j, j'. Notice that the matrix above obeys this condition.

The trivial bound on the number of ones in a matrix is $n^2 = n^{4/2}$ because that is a matrix where all values are one. We will show in the following section that a stronger bound can be found for a matrix with our "rectangle" constraint. We can improve our exponent by $1/2$ and find that a matrix can contain at most $Cn^{3/2}$ many ones where C is a constant.

2 The Two Dimensional Box Problem

Theorem 2.1. *The Two Dimensional Box Problem*
*Consider an n by n matrix of 1's and 0's such that I_{ij} * $I_{ij'}$ = 1 for at most one value of i. This matrix can contain at most $Cn^{3/2}$ many ones where C is a constant.*

Proof. Our matrix I contains

$$\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij}$$

many points where I_{ij} * $I_{ij'}$ = 1 for at most one value of i. Thus our goal is to bound this sum. We know

$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij}\Big)^2 = \Big(\sum_{i=1}^{n}\big(\sum_{j=1}^{n} I_{ij}\big) * 1\Big)^2$$

Applying the Cauchy Schwartz Inequality

$$\leq \sum_{i=1}^{n} 1^2 * \sum_{i=1}^{n}\big(\sum_{j=1}^{n} I_{ij}\big)^2$$

$$= n\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{j'=1}^{n} I_{ij}I_{ij'}$$

$$\leq n * \sum_{i=1}^{n}\sum_{j=j'}^{n} I_{ij}I_{ij'} + n * \sum_{i=1}^{n}\sum_{j\neq j'}^{n} I_{ij}I_{ij'}$$

Let us call this first value in the sum $\mathbb{A}$ and the other $\mathbb{B}$ such that

$$\mathbb{A} = n\sum_{i=1}^{n}\sum_{j=j'}^{n} I_{ij}I_{ij'}$$

$$= n\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij}^2$$

$$= n\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij} \leq n^3$$

so $\mathbb{A} \leq n^3$

$$\mathbb{B} = n * \sum_{i=1}^{n}\sum_{j\neq j'}^{n} I_{ij}I_{ij'}$$

$$\leq n * n(n-1)$$

because the sum is less than the number of pairs (j,j') where j≠j' so $\mathbb{B} \leq n^3$

Thus

$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij}\Big)^2 \leq \mathbb{A} + \mathbb{B} \leq 2n^3$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij} \leq \sqrt{2}n^{3/2}$$

thus $\sum_{i=1}^{n}\sum_{j=1}^{n} I_{ij} \leq Cn^{3/2}$

□

This bound is sharp. Nets Katz, Elliot Krop, and Mauro Maggioni [KKM] discuss an example of sharpness in their paper "Remarks on the Box Problem." Consider an $q-1$ x $q-1$ matrix of ones and zeros following the "no rectangle" criterion where $q = p^{2n}$ for some p prime. This represents the incidences between points in $\mathbb{F}_q^2/\{(0,0)\}$ and lines of the form $ax + by = 1$ in $\mathbb{F}_q^2$ such that $(a.b) \in \mathbb{F}_q^2/\{(0,0)\}$. Two lines can only intersect at no more than one point. Thhe number of lines possible is the number of points in $\mathbb{F}_q^2/\{(0,0)\}$, which is (q^2-1). There are q points per line. Thus we can produce a matrix with $q(q^2-1)$ many ones, the number of lines possible multiplied by the number of points per line, which is close to the most possible..;l.,

3 The Box Problem in Three Dimensions

Similar to how in two dimensions the matrix is analogous to incidences between lines and points, in three dimensions the ones in the matrix represent where a point and plane coincide. Since two different planes intersect each other at no more than one line in $\mathbb{R}^3$, we cannot have a "cube" of ones. ie

$$\prod_{j_1}^{2} \cdots \prod_{j_d}^{2} I_{i_{1_{j_1}} \dots i_{d_{j_d}}} = 0 \ \ \forall i_{l_1} \neq i_{l_2}, l = 1, 2, ..., d$$

Although we will not prove it here, the number of ones is at most $cn^{11/4}$ where c is a constant.

[KKM] also discuss the three dimensional case of the box problem. They prove the following theorem to show the sharpness of the bound in three dimensions:

Theorem 3.1. *Let p be any prime. There is a* $p^3 - 1 \times p^3 - 1 \times p^3 - 1$ *tensor I containing* $p^2(p^3 - 1)^2$ *many ones that satisfies the condition*

$$\prod_{l=1}^{2} \prod_{m=1}^{2} \prod_{n=1}^{2} I_{i_l j_m k_n} = 0$$

The proof of this theorem is analogous to the proof of sharpness presented in two dimensions. Instead of considering lines, we examine planes in $\mathbb{F}_p^3$, which can be written as $ax + by + cz = 1$ where $(a, b, c) \in \mathbb{F}^3/\{(0, 0, 0)\}$. $\mathbb{F}_{p^3}$ is an extension of $\mathbb{F}_p$ by an irreducable cubic, ie an element of $\mathbb{F}_{p^3}/\{0\}$ can be written as $\alpha = a + rb + r^2c$. Thus we can say that P_α is the plane $ax + by + cz = 0$ when $\alpha = a + rb + r^2c$.

Consider the matrix I where $I_{ijk} = 1$ iff $k \in P_{i*j}$ and is 0 otherwise. I has $p^2(p^3 - 1)^2$ many ones, which is the numner of points per plane multiplied by the number of planes possible. This matrix satisfies the criterion that

$$\prod_{j_1}^{2} \cdots \prod_{j_d}^{2} I_{i_{1_{j_1}} \dots i_{d_{j_d}}} = 0$$

because the cardinality of

$$A_{ii'jj'} = P_{i*j} \cap P_{i'*j} \cap P_{i*j'} \cap P_{i'*j'}$$

is at most one due to the fact that two different planes are either parallel, intersect at a line, or intersect at a point.

4 The d Dimensional Box Problem

While the geometric interpretation of our matrix of ones and zeros is less clear in greater than three dimensions, it is still possible to define a limiting condition as we did in two and three dimensions and to find a bound for the number of ones possible in the resulting matrix. We will show this in the following theorem:

Theorem 4.1. *Consider a d-dimensional matrix of ones and zeros such that*

$$\prod_{j_1}^{2} \cdots \prod_{j_d}^{2} I_{i_{1_{j_1}} \ldots i_{d_{j_d}}} = 0 \;\; \forall i_{l_1} \neq i_{l_2}, l = 1, 2, ..., d$$

The number of ones in such a matrix is less than $Cn^{d-2^{-d+1}}$ where C is a constant

Proof. The number of ones in a d-dimensional matrix of ones and zeros is

$$\sum_{i_1, \ldots, i_d = 1}^{n} I_{i_1 \ldots i_d} = \sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d = 1}^{n} I_{i_1 \ldots i_d}$$

To bound this sum, we consider

$$\Big(\sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d = 1}^{n} I_{i_1 \ldots i_d} \Big)^2$$

$$\leq n^{d-1} * \sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d}^{n} \sum_{i'_d}^{n} I_{i_1 \ldots i_d} I_{i_1 \ldots i'_d}$$

$$\leq n^{d-1} * \sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d \neq i'_d}^{n} I_{i_1 \ldots i_d} I_{i_1 \ldots i'_d} + n^{d-1} * \sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d = i'_d}^{n} I_{i_1 \ldots i_d} I_{i_1 \ldots i'_d}$$

By Cauchy Schwartz.

Calling the first term of the sum $\mathbb{A}_d$ and the second $\mathbb{B}_d$, we see that

$$\mathbb{A}_d = n^{d-1} * \sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d \neq i'_d}^{n} I_{i_1 \ldots i_d} I_{i_1 \ldots i'_d}$$

$$\mathbb{A}_d^2 = n^{2(d-1)} * \Big(\sum_{i_1, \ldots, i_{d-1} = 1}^{n} \sum_{i_d \neq i'_d}^{n} I_{i_1 \ldots i_d} I_{i_1 \ldots i'_d} \Big)^2$$

$$\leq n^{2(d-1)} * n^d * \sum_{i_1,\ldots,i_{d-2},i_d,i'_d=1}^{n} \sum_{i_{d-1},i'_{d-1}}^{n} I_{i_1\ldots i_{d-1}i_d} I_{i_1\ldots i_{d-1}i'_d} I_{i_1\ldots i'_{d-1}i_d} I_{i_1\ldots i'_{d-1}i'_d}$$

$$\leq n^{2(d-1)+d} * \sum_{i_1,\ldots,i_{d-2},i_d,i'_d=1}^{n} \sum_{i_{d-1}\neq i'_{d-1}}^{n} I_{i_1\ldots i_{d-1}i_d} I_{i_1\ldots i_{d-1}i'_d} I_{i_1\ldots i'_{d-1}i_d} I_{i_1\ldots i'_{d-1}i'_d}$$

$$+n^{2(d-1)+d} * \sum_{i_1,\ldots,i_{d-2},i_d,i'_d=1}^{n} \sum_{i_{d-1}=i'_{d-1}}^{n} I_{i_1\ldots i_{d-1}i_d} I_{i_1\ldots i_{d-1}i'_d} I_{i_1\ldots i'_{d-1}i_d} I_{i_1\ldots i'_{d-1}i'_d}$$

By Cauchy Schwartz.

As before, we name the first term in the sum $\mathbb{A}_{d-1}$ and the second $\mathbb{B}_{d-1}$ so that

$$\mathbb{A}_{d-1}^2$$

$$\leq n^{(d-1)2^2+2d+d+1} \sum_{i_1,\ldots,i_{d-3},i_{d-1},i'_{d-1},i_d,i'_d=1}^{n} \sum_{i_{d-2}\neq i'_{d-2}}^{n} I_{i_1\ldots i_{d-2}i_{d-1}i_d} \ldots I_{i_1\ldots i'_{d-2}i'_{d-1}i_d} I_{i_1\ldots i'_{d-2}i'_{d-1}i'_d}$$

$$+n^{(d-1)2^2+2d+d+1} * \sum_{i_1,\ldots,i_{d-3},i_{d-1},i'_{d-1},i_d,i'_d=1}^{n} \sum_{i_{d-2}=i'_{d-2}}^{n} I_{i_1\ldots i_{d-2}i_{d-1}i_d} \ldots I_{i_1\ldots i'_{d-2}i'_{d-1}i_d} I_{i_1\ldots i'_{d-2}i'_{d-1}i'_d}$$

Continuing in this fashion and repeatedly applying Cauchy Schwartz we find that for $l = 2, 3, ..., d-1, d$

$$\mathbb{A}_l = n^{(d-1)2^{d-l}+d2^{d-l-1}+\cdots+(l+2)2+(l+1)} \sum_{i_1,\ldots i_{l-1},i_{l+1},i'_{l+1},\ldots,i_d,i'_d}^{n} \sum_{i_l\neq i'_l}^{n} I_{i_1\ldots i_l\ldots i_{d-2}i_{d-1}i'_d} \ldots I_{i_1\ldots i'_l\ldots i'_{d-2}i'_{d-1}i_d}$$

so

$$\mathbb{A}_2 = n^{(d-1)2^{d-2}+d2^{d-3}+\cdots+(4)2+(3)} \sum_{i_1,i_3,i'_3,\ldots,i_d,i'_d}^{n} \sum_{i_2\neq i'_2}^{n} I_{i_1i_2\ldots i_{d-2}i_{d-1}i'_d} \ldots I_{i_1i'_2\ldots i'_{d-2}i'_{d-1}i_d}$$

$$\leq n^{(d-1)2^{d-2}+d2^{d-3}+\cdots+(4)2+(3)} * n^{2(d-1)+1}$$

because $I_{i_1i_2\ldots i_{d-2}i_{d-1}i_d} I_{i_1\ldots i_{d-2}i_{d-1}i'_d} \ldots I_{i_1i'_2i'_3\ldots i'_{d-2}i'_{d-1}i'_d} = 1$ for at most one value of i_1.

Assuming for now that $\mathbb{B}_l \leq c\mathbb{A}_l$for c constant and $l = 2, 3, ...d-1, d$ (we will show this is true later) we find that

$$\mathbb{A}_d \leq (n^{(d-1)2^{d-2}+d2^{d-3}+\cdots+(4)2+(3)} * n^{2(d-1)+1})^{2^{-(d-2)}}$$

which is on the order of $n^{d-2^{-d+1}}$

Now we want to show that $\mathbb{B}_l \leq c\mathbb{A}_l$ for c constant and $l = 2, ...d-1, d$.

$$\mathbb{B}_l = n^{(d-1)2^{d-l}+d2^{d-l-1}+\cdots+(l+2)2+(l+1)} \sum_{i_1,\ldots i_{l-1},i_{l+1},i'_{l+1},\ldots,i_d,i'_d}^{n} \sum_{i_l=i'_l}^{n} I_{i_1\ldots i_l\ldots i_{d-2}i_{d-1}i'_d} \ldots I_{i_1\ldots i'_l\ldots i'_{d-2}i'_{d-1}i_d}$$

$$\leq n^{(d-1)2^{d-l}+d2^{d-l-1}+\cdots+(l+2)2+(l+1)} * n^{2d-l}$$

so $\mathbb{B}_l \leq c\mathbb{A}_l$ for c constant

$\square$

5 Two Dimensional Box Game

It is possible to turn the two dimensional box problem into a game with two players. Each player takes a turn choosing a spot in the matrix to place a one. The player who places a one in spot I_{ij} where $I_{ij}I_{ij'}I_{i'j}I_{i'j'} = 1$ loses.

For a 2x2 matrix the game is trivial, with the first player always winning regardless of strategy. For a 3x3 matrix the game is just slightly more complicated, with the first player always winning if they use a rational strategy. Finding strategies for nxn matricies for n>3 becomes more difficult and is an avenue for future research.

The following Java program allows players to play on a computer with a square matrix of arbitrary size. This program contains two classes. The class GamePanel contains the methods required for gameplay. The class SquareMatrixGame contains the code needed to set up the user interface.

```java
import java.awt.*;
import java.util.ArrayList;
import javax.swing.*;
import java.awt.event.*;


public class GamePanel extends JPanel {
    private JTextField[][] boxes = new JTextField[SquareMatrixGame.getN()][SquareMatrixGame.getN()];
    boolean done = false;
    int playernum = -1;

    public GamePanel(){
        setLayout(new GridLayout(SquareMatrixGame.getN(),SquareMatrixGame.getN()));
        setBackground(new Color(230, 255, 255) );

      for(int i=0 ; i<boxes.length ; i++){
          for(int j=0 ; j<boxes[i].length ; j++){
              boxes[i][j] = new JTextField(10);
              boxes[i][j].setText("0");
              boxes[i][j].addActionListener(new MatrixListener());
              add(boxes[i][j]);
          }
      }
    }

    public void paintComponent(Graphics page){
        super.paintComponent(page);
    }
     //game play methods

     public static ArrayList<Integer> multiples(int[] column, int index){
         ArrayList<Integer> positions = new ArrayList<Integer>();
         for(int i=index; i<column.length; i++){
             if(column[i] != 0){
                 positions.add(i);
             }
         }
         return positions;
     }


     public static boolean gameOver(int[][] matrix){
         boolean over = false;

         ArrayList<Integer> nonzero = new ArrayList<Integer>();
         for(int i=0 ; i<matrix.length ; i++){
             nonzero = multiples(matrix[i],0);
             for(int j = 0 ; j<nonzero.size(); j++){
                 for(int k=i+1 ; k<matrix.length ; k++){
                     for(int l = 0 ; l<nonzero.size(); l++){
                         if(matrix[k][nonzero.get(j)]*matrix[k][nonzero.get(l)] > 0 && l!=j){
                             over = true;
                             return over;
                         }
                     }
                 }
             }
         }
```

```java
        return over;
    }

    //getters and setters

    public boolean getDone(){
        return done;
    }

    public int getPlayernum(){
        return playernum;
    }

    //listeners

    private class MatrixListener implements ActionListener{
        public void actionPerformed (ActionEvent e){
            int[][] numbers = new int[SquareMatrixGame.getN()][SquareMatrixGame.getN()];
             for(int i=0 ; i<numbers.length ; i++){
                  for(int j=0 ; j<numbers[i].length ; j++){
                      numbers[i][j] = Integer.parseInt(boxes[i][j].getText());
                  }
             }
             playernum = ((playernum + 1) % SquareMatrixGame.getNumplayers());
             int player = playernum+1;
             done = gameOver(numbers);
             if(done){
                 JOptionPane.showMessageDialog(null, "Player " + player + " loses");
             }
        }
    }
}


public class SquareMatrixGame {

    public static int n = 4;
    public static int numplayers = -1;

    public static void main(String[] args) {

        n = Integer.parseInt(JOptionPane.showInputDialog("How many rows would you like?"));
        numplayers = Integer.parseInt(JOptionPane.showInputDialog("How many players are there?"));

        JFrame frame = new JFrame("Game");
        frame.setDefaultCloseOperation(JFrame.EXIT_ON_CLOSE);

        GamePanel game = new GamePanel();

        frame.getContentPane().add(game);
        frame.pack();
        frame.setVisible(true);
    }

    public void setN(int m){
        n = m;
    }

    public static int getN(){
        return n;
    }

    public static int getNumplayers(){
        return numplayers;
    }

}
```

References

[KKM] Katz, Nets Hawk, Elliot Krop, and Mauro Maggioni. "Remarks on the box problem." Mathematical Research Letters 9.4 (2002): 515-520.