LATTICE POINTS INSIDE RANDOM ELLIPSOIDS

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Abstract. Let $N_a(t) = \#\{t\Omega_a \cap \mathbb{Z}^d\}$, with $\Omega_a = \left\{(a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \dots a_d^{-\frac{1}{2}}x_d) : x \in \Omega\right\}$, where Ω is the unit ball. Let $E_a(t) = N_a(t) - t^d|\Omega_a|$. We give a simple proof of the fact that

$$(*) \qquad \left(\int_{\frac{1}{2}}^{2}\int_{\frac{1}{2}}^{2}\cdots\int_{\frac{1}{2}}^{2} |E_a(t)|^2 da_1 da_2 \dots da_d\right)^{\frac{1}{2}} \lessapprox t^{\frac{d-1}{2}}$$

in 2 and 3 dimensions.

Introduction

Let

$$N_a(t) = \#\{t\Omega_a \cap \mathbb{Z}^d\}, \tag{0.1}$$

where

$$\Omega_a = \left\{(a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \dots, a_d^{-\frac{1}{2}}x_d) : x \in \Omega\right\}, \tag{0.2}$$

with $\frac{1}{2} \leq a_j \leq 2$, where Ω is the unit ball.

Let

$$N_a(t) = t^d|\Omega_a| + E_a(t). \tag{0.3}$$

A classical result due to Landau says that

$$|E_a(t)| \lesssim t^{d-2+\frac{2}{d+1}}, \tag{0.4}$$

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where here and throughout the paper, $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$. Similarly, $A \lessapprox B$, with a parameter t, means that given $\delta > 0$ there exists $C_\delta > 0$ such that $A \le C_\delta t^\delta B$.

A number of improvements over (0.4) have been obtained over the years in two and three dimensions. The best known result in three dimensions, to the best of our knowledge is $|E_a(t)| \lessapprox t^{\frac{21}{16}}$ proved by Heath-Brown ([H-B97]), improving on an earlier breakthrough due to Vinogradov ([Vinograd63]). It is proved by Szego in [Szego26] that

$$\left| E_{1,1,1}(t) - \frac{4\pi}{3} t^3 \right| \gtrsim t \log(t). \tag{0.5}$$

In two dimensions, the best known result is $|E_a(t)| \lessapprox t^{\frac{46}{73}}$ due to Huxley ([Huxley96]). A classical result due to Hardy says that

$$\left| E_{1,1}(t) - \pi t^2 \right| \gtrsim t^{\frac{1}{2}} \log^{\frac{1}{2}}(t). \tag{0.6}$$

Thus it is reasonable to conjecture that the estimate

$$|E_a(t)| \lessapprox t^{\frac{d-1}{2}} \tag{0.7}$$

holds in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In higher dimensions, the problem of point-wise estimate of $E_a(t)$ is completely solved. It is a result of Walfisch that if $d \ge 4$, then $|E_a(t)| \lessapprox t^{d-2}$, and logarithm may be removed in dimension 5 and greater. It is also known that if the eccentricities $(a_1, \dots, a_d)$ are rational, then this estimate is essentially sharp.

It is not known if there exists a single $a = (a_1, a_2, \dots, a_d)$ such that $|E_a(t)| \lessapprox t^{\frac{d-1}{2}}$ in any dimension. The question of finding such an a was posed by Sarnak in a two-dimensional setting a number of years ago. Sarnak's question would be answered by the following estimate.

Conjecture 0.2. *Given any $\delta > 0$,*

$$\sup_{t \ge 1} t^{-\frac{d-1}{2} - \delta} |E_{(\cdot)}(t)| \in L^p \left(\left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right] \times \dots \times \left[\frac{1}{2}, 2\right] \right), \tag{0.8}$$

for some $p \ge 1$ with a constant depending on δ.

In fact, (0.8) would, of course, imply that the estimate $|E_a(t)| \lessapprox t^{\frac{d-1}{2}}$ holds for almost every $a \in \left(\left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right] \times \dots \times \left[\frac{1}{2}, 2\right] \right)$. We hope to address this issue in a subsequent paper.

Other types of square averages of lattice point discrepancy functions have been studied in the past and in recent years. For example, a classical result due to Kendall says that

$$\int_{\mathbb{T}^2} \left| \#\{(t\Omega + \tau) \cap \mathbb{Z}^d\} - t^d |\Omega| \right|^2 d\tau \lesssim t^{\frac{d-1}{2}}, \tag{0.9}$$

for every convex domain where the boundary has everywhere non-vanishing Gaussian curvature.

This result was recently sharpened up by Magyar and Seeger. They proved that the estimate (0.9) still holds in $\mathbb{R}^d$ if the exponent 2 is replaced by $p \leq \frac{2d}{d-1}$.

Another type of average is studied in [ISS02]. The authors prove that

$$\left(\frac{1}{h}\int_R^{R+h} \left|\#\{t\Omega \cap \mathbb{Z}^d\} - t^d|\Omega|\right|^2 dt\right)^{\frac{1}{2}} \lesssim R^{\alpha_d}, \tag{0.10}$$

where

$$\alpha_2 = \frac{1}{2}, \text{ with } h \geq \log(R), \tag{0.11}$$

and

$$\alpha_d = d-2, \text{ with } h \approx R, \tag{0.12}$$

for $d \geq 4$. When $d = 3$, $\alpha_d = 1$ and an additional factor $\log(R)$ is present. These results improve results previously obtained by Muller ([Muller97]). See also [Huxley96] and [ISS02] and references contained therein.

Using (0.10), (0.11), and (0.12) and their proofs one can deduce the following result.

Theorem 0.1. *Let $E_a(t)$ be as above. Then*

$$\int_{\frac{1}{2}}^{2}\int_{\frac{1}{2}}^{2}\cdots\int_{\frac{1}{2}}^{2} |E_a(t)|^2 da \lesssim R^{\alpha_d}, \tag{0.13}$$

where α_d is exactly as above, and the additional $\log(t)$ factor is still present in three dimensions.

The purpose of this paper is to give a simple and transparent proof of Theorem 0.1 in two and three dimensions. Similar two-dimensional results have recently been obtained by different methods by Toth and Petridis in [TothPetridis02]. We believe that it is likely that our approach will lead to a better estimate in higher dimensions where we conjecture that (0.13) holds with $\alpha_d = \frac{d-1}{2}$. We hope to address this issue in a subsequent paper.

SECTION I: PROOF OF THEOREM 0.1 IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

We shall give a proof in three dimensions. We shall then indicate how a two-dimensional proof follows from a simpler version of the same argument.

We start with the following standard reduction. Let $\rho_0 \in C_0^\infty\left(\frac{1}{4}, 4\right)$ with $\rho_0 \equiv 1$ on $[1,2]$, and let ρ be the radial extension of ρ_0 such that $\int \rho(x)dx = 1$.

$\rho_\epsilon(x) = \epsilon^{-3}\rho\left(\frac{x}{\epsilon}\right)$. Let

$$N_a^\epsilon(t) = \sum_{k\in\mathbb{Z}^3} \chi_{t\Omega_a} * \rho_\epsilon(k) = t^3|\Omega_a| + t^3 \sum_{k\neq(0,0,0)} \widehat{\chi}_{\Omega_a}(tk)\widehat{\rho}(\epsilon k) = t^3|\Omega_a| + E_a^\epsilon(t). \tag{1.1}$$

It is not hard to see that there exists $C > 0$ such that

$$N_a^\epsilon(t - C\epsilon) \leq N_a(t) \leq N_a^\epsilon(t + C\epsilon). \tag{1.2}$$

It follows that

$$\int_{\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]} |E_a(t)|^2 da \lesssim \int_{\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]\times\left[\frac{1}{2},2\right]} |E_a^\epsilon(t)|^2 da + t^4\epsilon^2. \tag{1.3}$$

We conclude that it suffices to establish estimates for $E_a^\epsilon(t)$ with $\epsilon = t^{-1}$.

Using the standard asymptotic formula for the Fourier transform of the characteristic function of a bounded smooth convex domain where the Gaussian curvature of the boundary is non-vanishing, (see e.g [Hertz62]), we see that $\widehat{\chi}_{\Omega_a}(tk)$ is a sum of two terms of the form

$$e^{2\pi it|k|_a}\ t^{-2}|k|_a^{-2} + O((t|k|)^{-3}), \tag{1.4}$$

where

$$|k|_a = \sqrt{a_1k_1^2 + a_2k_2^2 + a_3k_3^2}. \tag{1.5}$$

It follows that

$$E_a^\epsilon(t) = t\sum_{k\neq(0,0,0)} e^{2\pi it|k|_a}|k|_a^{-2}\widehat{\rho}(\epsilon k) + t^3\sum_{k\neq(0,0,0)} O((t|k|)^{-3})\widehat{\rho}(\epsilon k) = I + II. \tag{1.6}$$

Since we can easily handle II point-wise, we turn our attention to I. Squaring, integrating in a, and replacing the limits of integration in a by a smooth cutoff function, we get

$$t^2\sum_{k,l\neq(0,0,0)} |k|^{-2}|l|^{-2}\widehat{\rho}(\epsilon k)\widehat{\rho}(\epsilon l)\int e^{2\pi it(|k|_a - |l|_a)}\psi_{k,l}(a)da$$

$$= t^2\sum_{k,l\neq(0,0,0)} |k|^{-2}|l|^{-2}\widehat{\rho}(\epsilon k)\widehat{\rho}(\epsilon l)I_{k,l}(t) \tag{1.7}$$

where

$$\psi_{k,l}(a) = \left(\frac{|k|}{|k|_a}\right)^2\left(\frac{|l|}{|l|_a}\right)^2\psi(a), \tag{1.8}$$

where ψ is a positive smooth cutoff function, supported in $[1/4, 4]$ and identically equal to 1 on $[1/2, 2]$. Observe that when $k \neq (0,0,0)$ and $l \neq (0,0,0)$, $\psi_{k,l} \in C_0^\infty$ with constants uniform in k and l. It suffices to show that (1.7) is bounded above by $C_\delta t^{2+\delta}$ for any $\delta > 0$.

Section II: Preliminary reductions

This section contains some simple observations that we shall make use of in Section III where the main result of the paper is proved.

Lemma 2.1. *Let $\delta > 0$. Let $N > \frac{1}{\delta} + 1$. Then*

$$\sum_{|k| > \epsilon^{-1-\delta}} |k|^{-2} |\epsilon k|^{-N} \lesssim 1. \tag{2.1}$$

Proof of Lemma 2.1. We have

$$\sum_{|k| > \epsilon^{-1-\delta}} |k|^{-2} |\epsilon k|^{-N} \lesssim \epsilon^{-N} \int_{|x| > \epsilon^{-1-\delta}} |x|^{-2-N} dx$$

$$\lesssim \epsilon^{-N} \epsilon^{-1-\delta} \epsilon^{N} \epsilon^{\delta N} \frac{1}{N-1} \lesssim 1, \tag{2.2}$$

if $N > \frac{1}{\delta} + 1$.

Since $|\widehat{\rho}(\epsilon k)| \lesssim (1 + |\epsilon k|)^{-N}$ for any $N > 0$, and $|I_{k,l}(t)| \lesssim 1$, Lemma 2.1 shows that in estimating (1.7) we may sum over $|k|, |l| \lesssim \epsilon^{-1-\delta}$, $\delta > 0$. In particular, this means that we may sum over $|k_j|, |l_j| \lesssim \epsilon^{-1-\delta}$.

Lemma 2.2. *Let S, S' be subsets of $\{1, 2, 3\}$ of cardinality at most 2. Then*

$$t^2 \sum_{1 \leq |k_i|, |l_j| \lesssim \epsilon^{-1-\delta}; i \in S, j \in S'} |k|^{-2} |l|^{-2} \lessapprox t^2. \tag{2.3}$$

Proof of Lemma 2.2. The proof is immediate since we are down to at most 2 variables in k and l, so the power -2 sufficies, up to logarithms.

Lemma 2.3. *Let $U = \{k, l \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; k_1 = 0, l_1 \neq 0\}$. Then*

$$t^2 \sum_U |k|^{-2} |l|^{-2} I_{k,l}(t) \lessapprox t^2. \tag{2.4}$$

Proof of Lemma 2.3. Let $\Phi_{k,l}(a) = |k|_a - |l|_a$. We have

$$\nabla \Phi_{k,l}(a) = \frac{1}{2} \left(\frac{k_1^2}{|k|_a} - \frac{l_1^2}{|l|_a}, \frac{k_2^2}{|k|_a} - \frac{l_2^2}{|l|_a}, \frac{k_3^2}{|k|_a} - \frac{l_3^2}{|l|_a} \right). \tag{2.5}$$

Since $k_1 = 0$, $|\nabla \Phi_{k,l}(a)| \gtrsim \frac{l_1^2}{|l|}$. Integrating by parts once (see the appendix) shows that

$$|I_{k,l}(t)| \lesssim t^{-1} \frac{|l|}{l_1^2}. \tag{2.6}$$

We get

$$t^2 t^{-1} \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; k_1 = 0} |k|^{-2} |l|^{-2} |l| l_1^{-2}$$

$$\lessapprox t \sum_{1 \le |l_j| \lesssim \epsilon^{-1-\delta}} (|l_2| + |l_3|)^{-1} l_1^{-2} \lessapprox t \epsilon^{-1} \lesssim t^2. \tag{2.7}$$

The same argument works if $k_2 = 0$ and $l_2 \neq 0$, or if $k_3 = 0$ and $l_3 \neq 0$.

The basic idea of these reductions is that we only need to sum up to $|k|, |l| \lesssim \epsilon^{-1-\delta}$, and that it suffices to consider the case where $k_j, l_j \neq 0$, $j = 1, 2, 3$.

Section 3: $\left|\left|\frac{k_1}{k_2}\right| - \left|\frac{l_1}{l_2}\right|\right| + \left|\left|\frac{k_1}{k_3}\right| - \left|\frac{l_1}{l_3}\right|\right| + \left|\left|\frac{k_2}{k_3}\right| - \left|\frac{l_2}{l_3}\right|\right| \neq 0$

The determinant of the Hessian matrix of $\Phi_{k,l}$ with respect to (a_1, a_2) equals

$$-\frac{1}{16} \frac{(k_1^2 l_2^2 - k_2^2 l_1^2)^2}{|k|_a^3 |l|_a^3}, \tag{3.1}$$

and its absolute value is bounded from below by a constant multiple of

$$\frac{(k_1^2 l_2^2 - k_2^2 l_1^2)^2}{|k|^3 |l|^3}. \tag{3.2}$$

It follows that

$$t^2 \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; \left|\left|\frac{k_1}{k_2}\right| - \left|\frac{l_1}{l_2}\right|\right| \neq 0} |k|^{-2} |l|^{-2} I_{k,l}(t)$$

$$\lesssim t \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; \left|\left|\frac{k_1}{k_2}\right| - \left|\frac{l_1}{l_2}\right|\right| \neq 0} |k|^{-\frac{1}{2}} |l|^{-\frac{1}{2}} |k_1^2 l_2^2 - k_2^2 l_1^2|^{-1}$$

$$\lesssim t \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; \left|\left|\frac{k_1}{k_2}\right| - \left|\frac{l_1}{l_2}\right|\right| \neq 0} |k_3|^{-\frac{1}{2}} |l_3|^{-\frac{1}{2}} |k_1^2 l_2^2 - k_2^2 l_1^2|^{-1}$$

$$\lessapprox t\epsilon^{-1} \sum_{1\le |k_j|,|l_j|\lesssim \epsilon^{-1-\delta};j=1,2;\left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right|\neq 0} |k_1^2l_2^2-k_2^2l_1^2|^{-1}. \tag{3.3}$$

Either $sgn(k_1l_2) = sgn(l_1k_2)$ or $sgn(k_1l_2) = -sgn(l_1k_2)$. Without loss of generality suppose that $k_j, l_j > 0$. It follows that (3.3) is bounded by the expression of the form

$$t\epsilon^{-1} \sum_{m=0}^{\approx \log(\epsilon^{-2})} 2^{-m} \left| \sum_{1\le k_j,l_j\lesssim \epsilon^{-1-\delta},j=1,2;\ 2^m\le |k_1l_2-k_2l_1|\le 2^{m+1}} k_1^{-1}l_2^{-1} \right|$$

$$\lessapprox t\epsilon^{-1} \sum_{m=0}^{\approx \log(\epsilon^{-2})} 2^{-m} \left| \int_{1\le x_j,y_j\le \epsilon^{-1};2^m\le |x_1x_2-y_1y_2|\le 2^{m+1}} x_1^{-1}x_2^{-1}dxdy \right|. \tag{3.4}$$

Let

$$u_1 = x_1x_2,\ u_2 = x_2,\ v_1 = y_1y_2,\ \text{and } v_2 = y_2. \tag{3.5}$$

It follows that

$$du_1 = x_2dx_1 + x_1dx_2,\ du_2 = dx_2,\ dv_1 = y_2dy_1 + y_1dy_2,\ \text{and } dv_2 = dy_2. \tag{3.6}$$

Also, $x_1 = \frac{u_1}{u_2}$, so $x_1x_2 = u_1$. Combining this with (3.5) and (3.6), we see that (3.4) is bounded by

$$t\epsilon^{-1} \sum_{m=0}^{\approx \log(\epsilon^{-2})} 2^{-m} \left| \int_{1\le u_1,v_1\le \epsilon^{-2},1\le u_2,v_2\le \epsilon^{-1};2^m\le |u_1-v_1|\le 2^{m+1}} u_1^{-1}u_2^{-1}v_2^{-1}dudv \right|$$

$$\lessapprox t\epsilon^{-1} \sum_{m=0}^{\approx \log(\epsilon^{-2})} 2^{-m} \left| \int_{1\le u_1,v_1\le \epsilon^{-2};2^m\le |u_1-v_1|\le 2^{m+1}} u_1^{-1}du_1dv_1 \right| \lessapprox t\epsilon^{-1} \le t^2. \tag{3.7}$$

Clearly, the same argument works if $\left|\left|\frac{k_1}{k_3}\right|-\left|\frac{l_1}{l_3}\right|\right| \neq 0$ or if $\left|\left|\frac{k_2}{k_3}\right|-\left|\frac{l_2}{l_3}\right|\right| \neq 0$.

Section 4: $\left|\left|\frac{k_1}{k_2}\right|-\left|\frac{l_1}{l_2}\right|\right| + \left|\left|\frac{k_1}{k_3}\right|-\left|\frac{l_1}{l_3}\right|\right| + \left|\left|\frac{k_2}{k_3}\right|-\left|\frac{l_2}{l_3}\right|\right| = 0$

In this case

$$\left|\frac{k_1}{l_1}\right| = \left|\frac{k_2}{l_2}\right| = \left|\frac{k_3}{l_3}\right|. \tag{4.1}$$

It follows that $k = \alpha l$. Dominating $|I_{k,l}(t)|$ by 1, we have

$$t^2 \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; \left|\frac{k_1}{l_1}\right| = \left|\frac{k_2}{l_2}\right| = \left|\frac{k_3}{l_3}\right|} |k|^{-2} |l|^{-2} I_{k,l}(t). \tag{4.2}$$

We are summing over the set where $l = \alpha k$. Observe that α must be of the form $\frac{m}{gcd(k_1,k_2,k_3)}$. It follows that the expression in (4.2) is bounded by a constant multiple of

$$\lesssim t^2 \sum_{1 \le |k| \lesssim \epsilon^{-1-\delta}} \sum_{\alpha = \frac{m}{gcd(k_1,k_2,k_3)} \lesssim \epsilon^{-1-\delta}} \alpha^{-2} |k|^{-4}$$

$$= t^2 \sum_{1 \le |k| \lesssim \epsilon^{-1-\delta}} \sum_{m=1}^{\approx \frac{\epsilon^{-1-\delta}}{gcd(k_1,k_2,k_3)}} \frac{(gcd(k_1,k_2,k_3))^2}{m^2} |k|^{-4}$$

$$\lesssim t^2 \sum_{1 \le |k| \lesssim \epsilon^{-1-\delta}} (gcd(k_1,k_2,k_3))^2 |k|^{-4}$$

$$= t^2 \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} 2^{-4n} \sum_{|k| \approx 2^n} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} \sum_{gcd(k_1,k_2,k_3)=j} j^2$$

$$\approx t^2 \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} 2^{-4n} \sum_{|k| \approx \frac{2^n}{j}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} \sum_{gcd(k_1,k_2,k_3)=1} j^2$$

$$\lesssim t^2 \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-4n} \frac{2^{3n}}{j^3} j^2$$

$$= t^2 \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-n} j^{-1} \lessapprox t^2. \tag{4.3}$$

This completes the three dimensional proof. We now outline the two dimensional argument. The determinant of the Hessian matrix of $\Phi_{k,l}$ in two dimensions is given by (3.1). When $\left|\frac{k_1}{k_2}\right| \neq \pm \left|\frac{l_1}{l_2}\right|$, the calculation identical to the one contained in (3.3)-(3.7) does the job. If $\left|\frac{k_1}{k_2}\right| = \pm \left|\frac{l_1}{l_2}\right|$, we repeat the argument in (4.2), (4.3) as follows

$$t \sum_{1 \le |k_j|, |l_j| \lesssim \epsilon^{-1-\delta}; \left|\frac{k_1}{l_1}\right| = \left|\frac{k_2}{l_2}\right|} |k|^{-\frac{3}{2}} |l|^{-\frac{3}{2}} I_{k,l}(t)$$

$$\lesssim t \sum_{1\le |k_j|,|l_j|\lesssim \epsilon^{-1-\delta};\left|\frac{k_1}{l_1}\right|=\left|\frac{k_2}{l_2}\right|} |k|^{-1}|l|^{-1} I_{k,l}(t).$$

$$\lesssim t \sum_{1\le |k|\lesssim \epsilon^{-1-\delta}} \sum_{\alpha=\frac{m}{gcd(k_1,k_2)}\lesssim \epsilon^{-1-\delta}} \alpha^{-1}|k|^{-2}$$

$$= t \sum_{1\le |k|\lesssim \epsilon^{-1-\delta}} \sum_{m=1}^{\approx \frac{\epsilon^{-1-\delta}}{gcd(k_1,k_2)}} \frac{gcd(k_1,k_2)}{m}|k|^{-2}$$

$$\lesssim t \sum_{1\le |k|\lesssim \epsilon^{-1-\delta}} gcd(k_1,k_2)|k|^{-2}$$

$$= t \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} 2^{-2n} \sum_{|k|\approx 2^n} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} \sum_{gcd(k_1,k_2)=j} j$$

$$\approx t \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} 2^{-2n} \sum_{|k|\approx \frac{2^n}{j}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} \sum_{gcd(k_1,k_2)=1} j$$

$$\lesssim t \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-2n}\frac{2^{2n}}{j^2} j$$

$$= t \sum_{n=1}^{\approx \log(\epsilon^{-1-\delta})} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-n} j^{-1} \lessapprox t. \tag{4.4}$$

Appendix: Oscillatory integrals of the first kind

In this paper we made use of the following basic facts about the oscillatory integrals of the form

$$I(t)=\int_{\mathbb{R}^d} e^{itf(x)}\psi(x)dx, \tag{5.1}$$

where ψ is a smooth cutoff function and f is smooth. See, for example [Stein93], [BNW88] for related information.

Theorem 5.1. *Suppose that f is convex and finite type, and the hessian matrix of f contains an M by M sub-matrix of determinant $\ge c_0$. Then*

$$|I(t)| \lesssim t^{-\frac{M}{2}} c_0^{-\frac{1}{2}}. \tag{5.2}$$

Theorem 5.2. *Suppose that* $|\nabla f(a)| \gtrsim c_0$. *Then*

$$|I(t)| \lesssim t^{-1} c_0^{-1}. \tag{5.3}$$

We note that in both theorems the constants may depend on the upper bounds of derivatives of f and ψ.

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