MATH 265H, FALL 2022, HOMEWORK #4

ALEX IOSEVICH

1. Problems not from the book

Problem #1: Let f be a continuously differentiable function on $[a,b]$, $a<b$. Suppose that $f'(x) \geq 1$ for all $x \in [a,b]$ and assume that f' an increasing function on $[a,b]$. Let $R > 10^6$. Prove that

$$\left| \int_a^b e^{iRf(x)} dx \right| \leq \frac{C}{R}$$

for some constant C independent of R.

Hint: Observe that

$$\int_a^b e^{iRf(x)} dx = \int_a^b \frac{1}{iRf'(x)} \left(e^{iRf(x)}\right)' dx$$

and integrate by parts. At some point you will need to use the fact that f' is an increasing function.

Problem #2: Let f be a twice continuously differentiable function on $[a,b]$, $a<b$. Suppose that $f''(x) \geq 1$ for all $x \in [a,b]$. Let $R > 10^6$. Prove that

$$\left| \int_a^b e^{iRf(x)} dx \right| \leq \frac{C}{\sqrt{R}}$$

for some constant C independent of R.

Hint: Note that f' is increasing since $f''(x) \geq 1$. Find a point c where $f'(c)=0$. Prove that there is at most one such point. If there is no such point, you are already done by Problem #1, but you need to think that through carefully. Write

$$\int_a^b e^{iRf(x)} dx = \int_a^{c-\delta} e^{iRf(x)} dx + \int_{c-\delta}^{c+\delta} e^{iRf(x)} dx + \int_{c+\delta}^b e^{iRf(x)} dx = I + II + III,$$

where δ is very small and will be specified later.

Show that $II \leq 2\delta$ using the triangle inequality. Estimate I and III using Problem #1 and Mean Value Theorem.

Historical Note: Both Problem #1 and Problem #2 appear as homework exercises in Edmund Landau's Calculus book used at the University of Gottingen in the 1920s.

2. Problems from the book

Pages 44-46: problems 19, 20, 22, 29, 30

Page 78: problems 1, 2, 3.