MATH 265H, FALL 2022, HOMEWORK #10

ALEX IOSEVICH

1. Problems not from the book

Problem: Let p be an odd prime. Define $\chi(t) = e^{\frac{2\pi i t}{p}}$, where $i = \sqrt{-1}$. Prove that for any a with $1 \leq a \leq p-1$,

$$\left| \sum_{t=0}^{p-1} \chi(at^2) \right| = \sqrt{p}.$$

Hint: Observe that

$$\left| \sum_{t=0}^{p-1} \chi(at^2) \right|^2 = \sum_{t=0}^{p-1} \sum_{s=0}^{p-1} \chi(a(t^2 - s^2)).$$

Make a change of variables $u = t - s, v = t + s$ and **prove** that the sum on the right hand side becomes

$$\sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \chi(auv).$$

Now rewrite this sum as

$$\sum_{k=0}^{p-1} \chi(ak) \sum_{\{(u,v):uv=k\}} 1,$$

and try to find the final sequence of steps in this proof! This idea is originally due to Carl Friedrich Gauss.

2. Problems from the book

Chapter 5, problems 7,9,11,12,15,22