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Math 265H, Fall 2022 October 19

Definition: A metric space in which every Cauchy sequence converges is called complete.

Definition: $\{s_n\} \in \mathbb{R}$

a) monotonically increasing if $s_n \leq s_{n+1} \ \forall n$

b) " decreasing if $s_n \geq s_{n+1} \ \forall n$

Theorem: $\{s_n\}$ monotonic. Then $\{s_n\}$ converges iff it is bounded.

Proof: Suppose $s_n \leq s_{n+1}$ & let $s = \underbrace{\text{l.u.b.}(\text{range}(s_n))}_{\text{makes sense if } \{s_n\} \text{ is bounded.}}$

It follows that $s_n \leq s \quad (n = 1, 2, \ldots)$

For every $\epsilon > 0$, there is $N \ni$

$$s - \epsilon < s_N \leq s \qquad \text{since } s = \text{l.u.b}$$

$$\Rightarrow s_n \xrightarrow[n \to \infty]{} s$$

②

Infinite limits: $\{s_n\}$

For every M, $\exists N \ni n \geq N$ implies that $s_n \geq M$. Then we say that $s_n \to \infty$.

Similarly, if (same conditions) $s_n \leq M$, $s_n \to -\infty$.

Definition: Let E = set of subsequential limits of $\underbrace{\{s_n\}}_{\text{sequence}}$ plus, possibly, $\pm\infty$.

Let $s^* = \sup E$, $s_* = \inf E$

$\quad \| \qquad\qquad\qquad \|$

$\limsup s_n \qquad \liminf s_n$

Theorem: $\{s_n\}$ sequence of real numbers. Let E, s^* be as above. Then

a) $s^* \in E$

b) If $x > s^*$, $\exists N \ni n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number w/ properties a) and b).

The proof is just definition chasing

Examples:

a) $\{\alpha_n\}$ all rationals $\qquad \overline{\lim}\, \alpha_n = +\infty$

$\underline{\lim}\, \alpha_n = -\infty$

b) $$s_n = \frac{(-1)^n}{1 + \frac{1}{n}}$$

$\overline{\lim}\, s_n = 1, \quad \underline{\lim}\, s_n = -1$

c) $s_n \xrightarrow[n \to \infty]{} s$ iff $\overline{\lim}\, s_n = \underline{\lim}\, s_n = s$ ✓

Theorem: If $s_n \leq t_n$ for $n \geq N$, N fixed, then

$\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$ ✓

$\underline{\lim}\, s_n \leq \underline{\lim}\, t_n$

④

Theorem: a) $p>0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$

b) $p>0 \Rightarrow \lim_{n\to\infty} \sqrt[n]{p} = 1$

c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.   d) $p>0$, α real

$\Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

e) If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

Proof: a) Take $n > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$...

b) Let $x_n = \sqrt[n]{p} - 1$. By binomial theorem,

$1 + nx_n \leq (1+x_n)^n = p \Rightarrow 0 < x_n \leq \frac{p-1}{n} \to 0$

c) Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and (binomial)

$$n = (1+x_n)^n \geq \frac{n(n-1)}{2} x_n^2$$

$$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \to 0 \quad n \geq 2$$

d) k integer chosen $\ni$ $k > \alpha$, $k > 0$.

For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} >$$

$$\frac{n^k p^k}{2^k k!} \Longrightarrow 0 < \frac{n^\alpha}{(1+p)^n} <$$

$$\frac{2^k k!}{p^k} n^{\alpha-k} \underset{n\to\infty}{\Longrightarrow} 0$$

e) $\alpha = 0$ in d)

<u>Series:</u> a_n sequence of real numbers

$S_n = \sum_{k=1}^{n} a_k$ If $S_n \to s$ finite,

then $\sum_{k=1}^{\infty} a_k$ is said to converge.