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Math 265H, Fall 2022, December 7

Theorem: $\varphi \nearrow$ continuous $[A,B] \to [a,b]$.

Suppose $\alpha \nearrow$ on $[a,b]$ & $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Let $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$.

Then $g \in \mathcal{R}(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha. \quad (*)$$

<u>Proof</u>: To each partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0, \dots, y_n\}$ of $[A,B]$ $\ni$ $x_i = \varphi(y_i)$.

We have $U(Q, g, \beta) = U(P, f, \alpha)$,

$$L(Q, g, \beta) = L(P, f, \alpha)$$

Since $f \in \mathcal{R}(\alpha)$, P can be chosen so that $U(P, f, \alpha)$ & $L(P, f, \alpha)$ approximate $\int f\, d\alpha$.

Theorem 6.6 $\Rightarrow$ $g \in \mathcal{R}(\beta)$ & $(*)$ holds.

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Theorem: $f \in \mathcal{R}(\alpha)$ $\alpha(x) = x$ on $[a,b]$.
For $a \le x \le b$, put $F(x) = \int_a^x f(t)\,dt$.
Then F is continuous on $[a,b]$. If f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Proof: $f \in \mathcal{R}$, so f is bounded.
Suppose $|f(t)| \le M$, $a \le t \le b$.
If $a \le x < y \le b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \le M(y-x),$$

and continuity is established.

Suppose that f is continuous at x_0. Given $\epsilon > 0$, choose $\delta > 0 \ni |f(t) - f(x_0)| < \epsilon$ if $|t - x_0| < \delta$, $a \le t \le b$. Hence, if

$x_0 - \delta < s \le x_0 \le t < x_0 + \delta$ and

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$a \le s < t \le b$, Theorem 6.12 implies that

$$\left| \frac{F(t)-F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t [f(u) - f(x_0)]\,du \right|$$

$$< \epsilon.$$

It follows that $F'(x_0) = f(x_0)$ ✓

FTC: If $f \in \mathcal{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b] \ni F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Proof: Let $\epsilon > 0$ be given. Choose

$P = \{x_0, x_1, \dots, x_n\}$ of $[a,b] \ni$

$U(P,f) - L(P,f) < \epsilon$. By MVT, $\exists\, t_i \in [x_{i-1}, x_i]$, such that $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$, $i = 1, 2, \dots$

Thus, $\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b) - F(a)$.

By Theorem 6.7, $\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \epsilon$.

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Integration by parts: F, G differentiable on $[a,b]$, $F' = f \in \mathcal{R}$, and $G' = g \in \mathcal{R}$. Then

$$\int_a^b F(x) g(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x)\, dx$$

easy consequence of the previous result.

Vector-valued functions:

$f_1, \ldots, f_k$ real-valued on $[a,b]$;

$\mathbf{f} = (f_1, \ldots, f_k)$ $\quad$ If $\alpha \nearrow$,

$\mathbf{f} \in \mathcal{R}(\alpha)$ means that $f_j \in \mathcal{R}(\alpha)$;

$$\int_a^b \mathbf{f}\, d\alpha = \left(\int_a^b f_1\, d\alpha, \ldots, \int_a^b f_k\, d\alpha \right)$$

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Theorem: $f:[a,b] \to \mathbb{R}^k$. If $f \in \mathcal{R}(\alpha)$, $\alpha \nearrow$ on $[a,b]$,

then $|f| \in \mathcal{R}(\alpha)$, and

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha.$$

Proof: If $f_1, f_2, \ldots, f_k$ components of f, then

$|f| = (f_1^2 + \ldots + f_k^2)^{\frac{1}{2}} \in \mathcal{R}(\alpha)$ since $f_i^2 \in \mathcal{R}(\alpha)$ & $\sqrt{\cdot}$ is continuous.

Let $y_j = \int f_j \, d\alpha$, so $y = \int f \, d\alpha$ &

$|y|^2 = \sum y_j^2 = \sum y_j \int f_j \, d\alpha =$

$\int (\sum y_j f_j) \, d\alpha \leq |y| \int |f| \, d\alpha$

by Cauchy-Schwarz and Theorem 6.12.

This completes the proof.

Rectifiable curves:

Definition: A continuous mapping $\gamma : [a,b] \to \mathbb{R}^k$ is called a curve in $\mathbb{R}^k$.

If γ is 1-1, γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is said to be a closed curve.

To each partition $P = \{x_0, x_1, \dots, x_n\}$ we associate

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

over all partitions

If $\Lambda(\gamma) < \infty$, γ is called rectifiable.

Theorem: If γ' is continuous on $[a,b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

Proof: If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right|$$

$$\le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt, \text{ so}$$

$$\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\,dt$$

for every partition P.

Consequently, $\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\,dt$

In the opposite direction, let $\epsilon > 0$ be given. Since γ' is uniformly continuous on $[a,b]$, there is $\delta > 0$ $\ni$ $|\gamma'(s) - \gamma'(t)| < \epsilon$ if $|s-t| < \delta$.

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$ w/ $\Delta x_i < \delta$ $\forall i$. If $x_{i-1} \le t \le x_i$, it follows that $|\gamma'(t)| \le |\gamma'(x_i)| + \epsilon$.

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Hence, $\int_{x_{i-1}}^{x_i} |\gamma'(t)|\,dt \leq |\gamma'(x_i)|\Delta x_i + \epsilon \Delta x_i$

$$= \left| \int_{x_{i-1}}^{x_i} \left[\gamma'(t) + \gamma'(x_i) - \gamma'(t)\right] dt \right| + \epsilon \Delta x_i$$

$$\overset{\text{Key}}{\leq} \left| \int_{x_{i-1}}^{x_i} \gamma'(t)\,dt \right| + \int_{x_{i-1}}^{x_i} \left[\gamma'(x_i) - \gamma'(t)\right] dt + \epsilon \Delta x_i$$

$$\leq \left| \gamma(x_i) - \gamma(x_{i-1}) \right| + 2\epsilon \Delta x_i.$$

Combining the inequalities yields

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(P, \gamma) + 2\epsilon(b-a)$$

$\leq \Lambda(\gamma) + 2\epsilon(b-a)$, which implies that

$$\int_a^b |\gamma'(t)|\,dt \leq \Lambda(\gamma).$$