Math 265 H, Fall 2022, November 30.

Theorem 6.9. If f is monotonic on $[a,b]$, and $\alpha \in C[a,b]$, then $f \in R(\alpha)$. (We still assume, of course, that α is monot.)

<u>Proof:</u> Let $\varepsilon > 0$ be given. For any $n \in \mathbb{N}$, choose a partition P_n s.t. $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ $\forall i \in \{1,..,n\}$;

This is possible by Intermediate Value Theorem since $\alpha \in C[a,b]$. WLOG, we assume that f is $\nearrow$ on $[a,b]$;

Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$ for $\forall i \in \{1,..,n\}$.

For any partition P_n, we have:

$$U(P_n,f,\alpha) - L(P_n,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}(M_i-m_i)$$

$$= \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1})) = \frac{\alpha(b)-\alpha(a)}{n}\cdot[f(b)-f(a)] < \varepsilon,$$

if n is taken large enough. For example, if

$n \in \mathbb{N}$: $\quad \frac{1}{n} < \frac{\varepsilon}{[\alpha(b)-\alpha(a)]\cdot[f(b)-f(a)]}$; Hence $f \in R(\alpha)$ $\blacksquare$

Theorem 6.10. Suppose f is bounded on $[a,b]$, f has finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in R(\alpha)$.

Proof: Let $\varepsilon > 0$ be given; Put $M = \sup\limits_{[a,b]} |f(x)|$;

Let E be the set of points at which f is discontinuous.

We know that E is finite and $\alpha \in C(E)$;

1) If $E = \emptyset$, then $f \in C[a,b] \Rightarrow f \in R(\alpha)$;

2) Assume that $E \neq \emptyset$; and $E = \{e_1, \dots, e_N\}$; Since $\alpha \in C(E) \Rightarrow$

$\forall i \in \{1, \dots, n\}$ $\exists \delta_i = \delta_i(\varepsilon) > 0$ $\forall t \in (e_i - \delta_i, e_i + \delta_i) \Rightarrow |\alpha(t) - \alpha(e_i)| < \frac{\varepsilon}{2N}$;

Hence we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a,b]$ s.t. $\sum\limits_{j=1}^{N} (\alpha(v_j) - \alpha(u_j)) < \varepsilon$;

Furthermore, we can place these intervals in such a way that any point of $E \cap (a,b)$ lies in the interior of some $[u_j, v_j]$. Remove the segments (u_j, v_j) from $[a,b]$. The remaining set K is compact.

Hence f is uniformly continuous on K, and $\exists \delta > 0$

$\forall s,t \in K: \ |s-t| < \delta \Rightarrow |f(s)-f(t)| < \varepsilon.$

Now form a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$, as follows:

Each u_j, v_j occurs in P. No point of any segment

(u_j, v_j) occurs in P. If x_{i-1} is not one of u_j, then

$\Delta x_i < \delta$. Note that $M_i - m_i \le 2M$ for $\forall i \in \{1, \ldots, n\}$. Also,

$M_i - m_i \le \varepsilon$, if $x_{i-1} \neq u_j$; Hence

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \le 2M \cdot \varepsilon + \varepsilon \cdot [\alpha(b) - \alpha(a)];$$

Since ε is arbitrary, it follows that $f \in R(\alpha)$. ∎

Theorem 6.11. Suppose $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$, $\phi \in C[m,M]$ and $h = \phi \circ f$ on $[a,b]$. Then $h \in R(\alpha)$ on $[a,b]$.

<u>Proof:</u> Let $\varepsilon > 0$ be given. Since $\phi \in UC[m,M]$, then $\exists \delta > 0$

such that $\delta < \varepsilon$, and $\forall s,t \in [m,M]: \ |s-t| < \delta \Rightarrow |\phi(s) - \phi(t)|$

$< \varepsilon$;

Since $f \in R(\alpha)$, then $\exists P = \{x_0, \ldots, x_n\}$ of $[a,b]$, s.t.

$$U(P,f,\alpha)-L(P,f,\alpha)<\delta^2.$$

Let $M_i = \sup_{[x_{i-1},x_i]} f$, $m_i = \inf_{[x_{i-1},x_i]} f$ and let

$M_i^* = \sup_{[x_{i-1},x_i]} h$ and $m_i^* = \inf_{[x_{i-1},x_i]} h$;

Let $\{1,\dots,n\} = A \sqcup B$, where: $A = \{i \in \{1,\dots,n\} \mid M_i - m_i < \delta\}$;

$B = \{i \in \{1,\dots,n\} \mid M_i - m_i \geqslant \delta\}$;

If $i \in A$, our choice of δ shows that $M_i^* - m_i^* \leq \varepsilon$;

Because $M_i^* - m_i^* = \sup_{p,q\in[x_{i-1},x_i]} |\phi(f(p)) - \phi(f(q))| \leq \varepsilon$ because

$|f(p)-f(q)| \leq M_i - m_i < \delta$;

If $i \in B$, then $M_i^* - m_i^* \leq 2K$, where $K = \sup_{[m,M]} |\phi(t)|$;

Hence

$$\delta \sum_{i\in B} \Delta\alpha_i \leq \sum_{i\in B} (M_i - m_i)\Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i =$$

$$= U(P,f,\alpha) - L(P,f,\alpha) < \delta^2 \Rightarrow$$

$$\sum_{i\in B} \Delta\alpha_i < \delta;$$

It follows that

$$U(P,h,\alpha)-L(P,h,\alpha)=\sum_{i=1}^{n}(M_i^*-m_i^*)\Delta\alpha_i=\sum_{i\in A}(M_i^*-m_i^*)\Delta\alpha_i+$$

$$+\sum_{i\in B}(M_i^*-m_i^*)\Delta\alpha_i\leq\delta\sum_{i\in A}\Delta\alpha_i+2K\sum_{i\in B}\Delta\alpha_i\leq$$

$$\leq\delta[\alpha(b)-\alpha(a)]+2K\delta<\varepsilon[\alpha(b)-\alpha(a)+2K];$$

Since ε was arbitrary it follows that $h\in R(\alpha)$ ■

Remark: The presence of even one discontinuity of the function $\phi(x)$ can lead to nonintegrability of the composition $\varphi\circ f$.

Indeed, let $\phi(x)=|\operatorname{sgn}(x)|=\begin{cases}1, & x\neq 0;\\ 0, & x=0;\end{cases}$

We can verify that if we take, say the Riemann function f on $[1,2]$, i.e. $f(x)=\begin{cases}1/q, & x=p/q,\ (p,q)=1;\\ 0, & x\notin\mathbb{Q};\end{cases}$

then the composition $\phi\circ f(x)$ on that interval is precisely the Dirichlet function $\mathcal{D}(x)$, where

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \notin \mathbb{Q}; \end{cases}$$

It is a really good exercise to show that Riemann function is integrable.

PROPERTIES OF THE INTEGRAL

Theorem 6.12

(a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \mathcal{R}(\alpha)$, $cf \in \mathcal{R}(\alpha)$ for every constant c, and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha,$$

$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha;$$

(b) If $f_1(x) \leq f_2(x)$ on $[a,b]$, and $f_1, f_2 \in \mathcal{R}(\alpha)$, then

$$\int_a^b f_1\, d\alpha \leq \int_a^b f_2\, d\alpha;$$

c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$, and

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha;$$

d) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \leq M$ on $[a,b]$, then

$$\left| \int_a^b f\,d\alpha \right| \leq M[\alpha(b) - \alpha(a)];$$

e) If $f \in \mathcal{R}(\alpha_1)$, $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$, and

$$\int_a^b f\,d(\alpha_1 + \alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2;$$

If $f \in \mathcal{R}(\alpha)$ and $c > 0$, then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha;$$