1

Math 265H, Fall 2022, November 28

$[a,b]$ interval $\quad f$ bounded on $[a,b]$

$P = x_0, x_1, \ldots, x_n \quad a = x_0 \leq x_1 \leq \ldots \leq x_n = b$

partition

$\Delta x_i = x_i - x_{i-1} \quad i = 1, 2, \ldots, n$

$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\overline{\int_a^b} f = \inf U(P, f)$$

$$\underline{\int_a^b} f = \sup L(P, f)$$

If they are equal, the integral is said to exist and is denoted by $\int_a^b f(x)\,dx$

Since $f$ is bounded, $\exists\ m, M \ni$

$m \leq f(x) \leq M$, so

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

So far, we have been "measuring length" using $\underline{dx}$. Let's up the level of sophistication.

$\alpha \nearrow$ on $[a,b]$

$\alpha(a), \alpha(b)$ $\underline{\underline{\text{finite}}}$

Let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i$$

Define $\overline{\int_a^b}$ & $\underline{\int_a^b}$ same way as before!

Definition: $P^*$ is a refinement of $P$

if $P^* \supset P$.

$P^*$ is a common refinement of $P_1, P_2$

if $P^* = P_1 \cup P_2$.

Theorem: If $P^*$ is a refinement of $P$,

then $L(P, f, \alpha) \leq L(P^*, f, \alpha)$

$U(P^*, f, \alpha) \leq U(P, f, \alpha)$

Proof: (Straightforward - work through it!)

Theorem: $\underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha$

Proof: $P^* = P_1 \cup P_2$

$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha)$

$\leq U(P_2, f, \alpha)$, so $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

④

Take sup over $P_1$, so

$$\underline{\int} f\,d\alpha \leq U(P_2, f, \alpha)$$

Now take sup over $P_2$. $\blacksquare$

Theorem: $f \in \mathcal{R}(\alpha)$ on $[a,b]$ iff for every $\epsilon > 0$ $\exists$ $P$ partition $\ni$

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon. \quad (*)$$

Proof: For every $P$ we have

$$L(P, f, \alpha) \leq \underline{\int} f\,d\alpha \leq \overline{\int} f\,d\alpha \leq U(P, f, \alpha)$$

$(*)$ implies that $0 \leq \overline{\int} f\,d\alpha - \underline{\int} f\,d\alpha < \epsilon$,

which implies that $\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha$,

so $f \in \mathcal{R}(\alpha)$.

Conversely, suppose $f \in \mathcal{R}(\alpha)$, and let $\epsilon > 0$ be given. Then $\exists$ $P_1, P_2$ partitions $\ni$

$$U(P_2, f, \alpha) - \int f\,d\alpha < \frac{\epsilon}{2},$$

(5)

$\int f\,d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2}.$

Let $P$ = common refinement of $P_1, P_2$

Then $U(P, f, \alpha) \leq U(P_2, f, \alpha) <$

$\int f\,d\alpha + \frac{\epsilon}{2} < L(P_1, f, \alpha) + \epsilon \leq L(P, f, \alpha) + \epsilon,$

and we are done.

We shall need a technical elaboration on this theme.

Theorem:

a) If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$, $(*)$ then the same holds for every refinement of $P$.

b) If $(*)$ holds for $P = \{x_0, x_1, \ldots, x_n\}$, and $s_i, t_i$ arbitrary in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

6

c) If $f \in \mathcal{R}(\alpha)$ and $t_i \in [x_{i-1}, x_i]$, then

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

<u>Proof:</u> a) is instant

b) $f(s_i), f(t_i) \in [m_i, M_i]$, so $|f(s_i) - f(t_i)| \le M_i - m_i$. Thus

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha),$$

so we are done.

c) This follows from

$$L(P, f, \alpha) \le \sum f(t_i)\Delta\alpha_i \le U(P, f, \alpha),$$

and $L(P, f, \alpha) \le \int f\,d\alpha \le U(P, f, \alpha)$

Theorem: If $f$ is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Proof: Let $\epsilon > 0$ be given. Choose $\eta > 0$ so that $[\alpha(b) - \alpha(a)]\eta < \epsilon$.

Since $f$ is uniformly continuous on $[a,b]$, $\exists\ \delta > 0 \ni$ $|f(x) - f(t)| < \eta$ if $|x - t| < \delta$.

Choose $P$ partition of $[a,b] \ni$ $\Delta x_i < \delta$ for all $i$, then implies that $M_i - m_i < \eta$, $(i = 1, 2, \dots, n)$, so

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$\leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \epsilon \checkmark$$

It follows that $f \in \mathcal{R}(\alpha)$.

8

Theorem: If $f$ is monotonic on $[a,b]$, and if $\alpha$ is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$

Proof: Let $\epsilon > 0$ be given. For any $n$, choose a partition $\ni$

$$\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}, \quad i = 1, 2, \ldots, n$$

Suppose that $f \nearrow$ (the other cases are the same)

Then $M_i = f(x_i)$, $m_i = f(x_{i-1})$ $(i=1,2,\ldots,n)$

so that

$$U(P,f,\alpha) - L(P,f,\alpha) = \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]$$

$$= \frac{\alpha(b)-\alpha(a)}{n}[f(b)-f(a)] < \epsilon \text{ if}$$

$n$ is large enough.