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Math 265H, Fall 2022, November 21

Theorem: f, g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq \infty$, Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$

If $f(x) \to 0$, $g(x) \to 0$ as $x \to a$,

or if $g(x), \cancel{f(x)} \to \infty$ as $x \to a$,

then $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

Proof: Let $-\infty \leq A < \infty$

Choose $q \ni A < q$ & $r \ni A < r < q$.

Then $\exists\ c \in (a,b) \ni a < x < c$ implies that

$$\frac{f'(x)}{g'(x)} < r.$$

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If $a < x < y < c$, then MVT $\Longrightarrow$

$\exists\ t \in (x, y) \ni$

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r. \qquad (*)$$

If $f(x), g(x) \to 0$ as $x \to a$, let $x \to a$ in $(*)$ and obtain

$$\frac{f(y)}{g(y)} \le r < q \quad (a < y < c) \quad \checkmark$$

Now suppose that $g(x) \to \infty$ as $x \to a$. Keep y fixed in $(*)$, choose $c_1 \in (a, y)$ $\ni g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Multiplying $(*)$ by $\frac{g(x)-g(y)}{g(x)}$, we

see that $\frac{f(x)}{g(x)} < r - \frac{r\, g(y)}{g(x)} + \frac{f(y)}{g(x)} \quad a < x < c_1$

③ Now let $x \to a$ and use the assumption $g(x) \to \infty$ as $x \to a$ to see that $\exists$ $c_2 \in (a, c_1)$ $\ni$ $\frac{f(x)}{g(x)} < q$ $(a < x < c_2)$.

Putting everything together, for any q $\ni$ $A < q$, $\exists\, c_2$ $\ni$ $\frac{f(x)}{g(x)} < q$ if $a < x < c_2$.

In the same way, if $-\infty < A \leq \infty$, and $p < A$, we can find c_3 $\ni$

$$p < \frac{f(x)}{g(x)} \quad (a < x < c_3)$$ and the proof is complete.

Taylor

Theorem: Suppose f is a real-valued function on $[a, b]$, n positive integer, $f^{(n-1)}$ continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points on $[a, b]$, and define

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$$p(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

Then $\exists$ x between α and β $\ni$

$$f(\beta)=p(\beta)+\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

Proof: When $n=1$, this is just MVT, so for higher n we must somehow reduce to that case.

Let M be given by

$$f(\beta)=p(\beta)+M(\beta-\alpha)^n$$ and let

$$g(t)=f(t)-p(t)-M(t-\alpha)^n \quad (a\le t\le b)$$

<u>Claim:</u> $n!\,M=f^{(n)}(x)$, for some $x\in(\alpha,\beta)$.

Differentiating g & plugging in the definition of p, we see that $g^{(n)}(t)=f^{(n)}(t)-n!\,M$, and we are left to show that $g^{n}(x)=0$ for some $x\in(\alpha,\beta)$

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Since $p^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k=0,1,\dots,n-1$,

we have $g(\alpha)=g'(\alpha)=\dots=g^{(n-1)}(\alpha)=0$

By definition of M, $g(\beta)=0$, so $g'(x_1)=0$ for some $\underbrace{x_1}_{(\alpha,\beta)}$ by MVT.

Since $g'(\alpha)=0$, we conclude that

$\exists\, x_2 \ni g''(x_2)=0$ for some $x_2 \in (\alpha, x_1)$.

Proceeding in this way, we see that $g^{(n)}(x_n)=0$ for some $x_n \in (\alpha, x_{n-1}) \subset (\alpha,\beta)$.

<u>Vector-valued functions:</u>

$f:[a,b] \longrightarrow \mathbb{R}^k$. Define

$$f' : \lim_{t\to x} \left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| = 0$$

$$f=(f_1,\dots,f_k), \quad f'=(f_1',\dots,f_k')$$

Things are difficult in higher dimensions:

Define $f(x) = e^{ix} = \cos x + i \sin x$

$f(2\pi) - f(0) = 1 - 1 = 0$, but

$f'(x) = ie^{ix}$, so $|f'(x)| \equiv 1$,

so MVT fails in this case.

More disasters: On $(0,1)$, define

$f(x) = x$ and $g(x) = x + x^2 e^{\frac{i}{x^2}}$.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

Also, $g'(x) = 1 + \left\{ 2x - \frac{2i}{x} \right\} e^{\frac{i}{x^2}}$ $(0 < x < 1)$

So $|g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1$

It follows that $\left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$

⑦

We conclude that $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0$,

and L'Hospital fails!

<u>But what is true?</u>

Theorem: $f : [a,b] \longrightarrow \mathbb{R}^k$ differentiable in (a,b).

Then $\exists\ x \in (a,b) \ni$

$$|f(b) - f(a)| \leq (b-a)|f'(x)|$$

Proof: Let $z = f(b) - f(a)$, and

$$\varphi(t) = z \cdot f(t) \qquad a \leq t \leq b$$

real-valued, continuous, differentiable on (a,b)

$$\text{MVT} \hookrightarrow \varphi(b) - \varphi(a) = (b-a)\,\varphi'(x) =$$

$$(b-a)\, z \cdot f'(x) \quad \text{for some } x \in (a,b).$$

also, $\varphi(b) - \varphi(a) = z \cdot f(b) - z \cdot f(a) = |z|^2$

$$|z|^2 = (b-a)\,|z \cdot f'(x)| \underset{\text{by Cauchy-Schwarz}}{\leq} (b-a)\,|z|\,|f'(x)|$$