Math 265, Fall 2022, November 16

Theorem: f continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval $\underbrace{f([a,b])}_{I}$, and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)), \quad a \leq t \leq b,$$

then h is differentiable at x, and

$$h'(x) = g'(f(x)) \cdot f'(x).$$

Proof: Let $y = f(x)$. By definition of the derivative, we have

$$f(t) - f(x) = (t-x)[f'(x) + u(t)]$$

$$g(s) - g(y) = (s-y)[g'(y) + v(s)],$$ where

$t \in [a,b]$, $s \in I$, and $u(t) \xrightarrow[t \to x]{} 0$,

(2)

$$\lim_{s \to y} v(s) \longrightarrow 0.$$

Let $s = f(t)$. Then

$$h(t) - h(x) = g(f(t)) - g(f(x))$$

$$= [f(t) - f(x)] \cdot [g'(y) + v(s)]$$

$$= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)].$$

If $t \neq x$,

$$\frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)]$$

$g'(y) f'(x)$ and we are done.

(3)

Mean-value theorem: f, g continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

Proof: Put

$$h(t) = [f(b)-f(a)]g(t) - [g(b)-g(a)]f(t)$$
$$(a \leq t \leq b)$$

Then h is continuous on $[a,b]$, h is differentiable in (a,b), and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b)$$

To prove the result, we just need to show that $h'(x) = 0$ for some $x \in (a,b)$.

If h is constant, there is nothing to prove.

④

If $h(t) > h(a)$ for some $t \in (a,b)$, let $x =$ the point where h attains its maximum. We claim (to be proved below) that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a,b)$, the same argument applies. Thus everything reduces to the following:

<u>Theorem:</u> f defined on $[a,b]$; if f has a local max $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

<u>Proof:</u> Choose $\delta \ni$

$$a < x-\delta < x < x+\delta < b$$

If $x-\delta < t < x$, then

$$\frac{f(t)-f(x)}{t-x} \geq 0 \Longrightarrow f'(x) \geq 0$$

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0, \text{ so } f'(x) \leq 0.$$

Hence, $f'(x) = 0$.

We can now deduce the classical formulation of MVT:

Theorem: If f is real-valued continuous on $[a, b]$ which is differentiable in (a, b), then $\exists\, x \in (a, b)$ at which $f(b) - f(a) = (b - a) f'(x)$

this is proved by taking $g(x) = x$ above!

Theorem: f differentiable on (a,b)

a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.

b) If $f'(x) = 0$ $\forall x \in (a,b)$, then f is constant.

c) If $f'(x) \leq 0$ $\forall x \in (a,b)$, then f is monotonically decreasing.

} follows from MVT.

<u>Continuity of derivatives:</u>

<u>Theorem:</u> f differentiable on $[a,b]$ and suppose that $f'(a) < \lambda < f'(b)$. Then $\exists\, x \in (a,b) \ni f'(x) = \lambda$.

Proof: Let $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so $g(t_1) < g(a)$ for some $t_1 \in (a,b)$, and $g'(b) > 0$, so $g(t_2) < g(b)$ for some $t_2 \in (a,b)$.

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Hence, g attains its min $\bar{x}$ on (a, b).

It follows that $g'(\bar{x}) = 0 \Leftrightarrow f'(x) = \lambda$ ✓

Limits using MVT:

$$\lim_{x \to 0} \frac{\ln(x+1)}{x} =$$

$$\lim_{x \to 0} \frac{\ln(x+1) - \ln(1)}{x} =$$

$0 < c(x) < 1$, say

$$\lim_{x \to 0} \frac{\cancel{x} \frac{1}{c(x)+1}}{\cancel{x}} = 1$$