Math 265H, Fall 2022, November 7

Corollary: $f: X \longrightarrow \mathbb{R}^k$

continuous

compact metric space

Then $f(X)$ is closed and bounded.

Corollary: $f: X \longrightarrow \mathbb{R}$

continuous

metric space

$M = \sup_{p \in X} f(p)$ $\quad$ $m = \inf_{p \in X} f(p)$

Then $\exists\, p, q \in X \ni f(p) = M,\ f(q) = m.$

Theorem: f continuous, 1-1 $\quad f: X \longrightarrow Y$

compact metric space $\quad$ metric space

Then f^{-1} is a continuous mapping from Y to X.

<u>Proof:</u> It is enough to show that $f(V)$ is open whenever V is open. Since X is compact, V^c is compact, so $f(V^c)$ is compact in Y, hence closed. Since f is 1-1, $f(V)$ is a

complement of $f(\overline{V}^c)$ & we are done.

<u>Definition:</u> $f: X \longrightarrow Y$ is <u>uniformly continuous</u>
metric spaces

if for every $\epsilon > 0$ $\exists$ $\delta > 0 \ni d_Y(f(p), f(q)) < \epsilon$
for all $p, q \ni d_X(p, q) < \delta$.

Example: $X = (0,1)$ $Y = (0,1)$

$$f(x) = \frac{1}{x}$$

Let's choose x_0 small and investigate the continuity at x_0. We want

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \epsilon \quad \text{if} \quad |x - x_0| < \delta$$

$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \frac{|x - x_0|}{|x| \cdot |x_0|}$, so if $|x - x_0| < \delta$

$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \frac{\delta}{|x| \cdot |x_0|}$. To make this

$<$ pre-assigned ϵ, $\frac{\delta}{|x| \cdot |x_0|} < \epsilon$, so δ <u>cannot</u>

③

be chosen independently of x_0.

Theorem: $f: X \longrightarrow Y$

f continuous

compact

Then f is uniformly continuous.

Proof: Let $\epsilon > 0$ be given. Hence $\exists\ \varphi(p) \ni$
$q \in X,\ d_X(p,q) < \varphi(p) \Longrightarrow d_Y(f(p), f(q)) < \frac{\epsilon}{2}$

Define $J(p) = \{q \in X : d_X(p,q) < \frac{1}{2}\varphi(p)\}$

Then $\{J(p)\}$ is an open cover of X which has a finite subcover by compactness, i.e

$$X \subset J(p_1) \cup \dots \cup J(p_n)$$

We put $\delta = \frac{1}{2}\min[\varphi(p_1), \dots, \varphi(p_n)]$. Let $q, p \in X$ $\ni d_X(p,q) < \delta$. Then $\exists\ p_m \ni d_X(p, p_m) < \frac{1}{2}\varphi(p_m)$

$< \frac{\delta}{\cancel{\ }}$

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Moreover, $d_X(p_m, q) \leq d_X(p, q) + d_X(p, p_m)$
$$< \delta + \tfrac{1}{2}\varphi(p_m) \leq \varphi(p_m)$$

It follows that
$$d_Y(f(p), f(q)) \leq d_X(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Theorem: $E \subseteq \mathbb{R}$, not compact. Then

a) $\exists f$ continuous on $\mathbb{R}$ which is not bounded.

b) $\exists$ continuous and bounded function on E which has no maximum.

c) If E is bounded, then $\exists$ continuous function on E which is not uniformly continuous.

Proof: Suppose that E is bounded, so $\exists$ limit point $x_0 \notin E$. Let

$$f(x) = \frac{1}{x - x_0}, \quad x \in E$$ continuous & unbounded!!!

The same function is not uniformly continuous as we discussed above!

Now consider $g(x) = \dfrac{1}{1+(x-x_0)^2}$ $x \in E$

continuous on E, and bounded, but $\sup_{x \in E} g(x) = 1$ is never achieved.

If E is unbounded, $f(x) = x$ yields a), and $h(x) = \dfrac{x^2}{1+x^2}$ yields b).

Continuity and connectedness:

Theorem: If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

Proof: Assume that $f(E) = A \cup B$, A, B separated and non-empty. Let $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, G, H non-empty.

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Since $A \subset \bar{A}$, $G \subset \underbrace{f^{-1}(\bar{A})}$

closed by continuity

It follows that $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B = \phi$, we conclude that $\bar{G} \cap H = \phi$, so G, H are separated. This is a contradiction since E is connected.

Intermediate Value Theorem

Corollary: $f : [a, b] \longrightarrow \mathbb{R}$ continuous.

If $f(a) < f(b)$ and $f(a) < c < f(b)$, then $\exists\, x \in (a, b) \ni f(x) = c$.

Proof: By Chapter 2, $[a, b]$ is connected, so $f([a, b])$ is connected, which implies the conclusion (Theorem 2.47), i.e. a subset E of $\mathbb{R}$ is connected iff $x < z < y$, $x, y \in E$ implies that $z \in E$.

Differentiation: For $x \in [a, b]$, define

$$\varphi(t) = \frac{f(t) - f(x)}{t - x}, \quad a < t < b,\ t \neq x$$

$$f'(x) = \lim_{t \to x} \varphi(t)$$

if it exists

<u>Theorem:</u> $f : [a, b] \to \mathbb{R}$. If f is differentiable at $x \in [a, b]$, then f is continuous at x.

Proof: $f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x)$

$$\xrightarrow{t \to x} f'(x) \cdot 0 = 0 \quad ✓$$

Theorem: f, g differentiable at $x \in [a, b]$

Then $f + g$, fg & f/g are differentiable at x, and

a) $(f + g)'(x) = f'(x) + g'(x)$ b)

b) $(fg)' = f'g + fg'$ c) $\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2}$