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Math 265H, Fall 2022, October 31

Power Series: $\{c_n\}$ complex numbers;

$$\underbrace{\sum c_n z^n}_{\text{coefficients}} \equiv \text{power series}$$

converges or diverges depending on $\underline{z}$

<u>Theorem</u>: Let $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$; $R = \frac{1}{\alpha}$

Then $\sum c_n z^n$ converges if $|z|<R$, and diverges if $|z|>R$.

<u>Proof</u>: Let $a_n = c_n z^n$. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = |z| \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \frac{|z|}{R},$$

and we are done by applying the root test.

$R \equiv$ radius of convergence

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Examples:

a) $\sum n^n z^n$ has $R=0$

b) $\sum \frac{z^n}{n!}$ has $R=\infty$

c) $\sum z^n$ has $R=1$.

d) $\sum \frac{z^n}{n}$ has $R=1$. The case $|z|=1$ is interesting and will be addressed separately.

e) $\sum \frac{z^n}{n^2}$ has $R=1$, and there are no convergence issues when $|z|=1$

__Summation by parts:__

$\{a_n\}$, $\{b_n\}$ sequences

$A_n = \sum_{k=0}^{n} a_k$, $n \geq 0$, $A_{-1}=0$. Then for $0 \leq p \leq q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \quad \checkmark$$

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Theorem: Suppose

a) $\{A_n\} = \{\sum_{k=1}^{n} a_k\}$ is a bounded sequence

b) $b_0 \geq b_1 \geq \dots \geq \dots$

c) $\lim_{n \to \infty} b_n = 0$

Then $\sum a_n b_n$ converges.

proof: Choose $M \ni |A_n| \leq M \ \forall n$.

Given $\epsilon > 0$, $\exists N \ni b_N \leq \frac{\epsilon}{2M}$.

For $N \leq p \leq q$,

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$$

$$\leq M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right| = 2M b_p \leq 2M b_N \leq \epsilon \checkmark$$

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Theorem: Suppose

a) $|c_1| \geq |c_2| \geq \dots$

b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$, $m = 1, 2, \dots$

c) $\lim_{n \to \infty} c_n = 0$

Then $\sum c_n$ converges

Proof: Take $a_n = (-1)^n$, $b_n = |c_n|$ in the previous theorem.

<u>Theorem:</u> Suppose that the radius of convergence of $\sum c_n z^n$ is $\underline{1}$; $c_0 \geq c_1 \geq c_2 \geq \dots$ and $\lim_{n \to \infty} c_n = 0$. Then $\sum c_n z^n$ converges for all z in $|z| = 1$, except possibly $z = \underline{1}$.

Proof: Let $a_n = z^n$, $b_n = c_n$. Observe that

$$|A_n| = \left| \sum_{m=0}^{n} z^m \right| = \left| \frac{1 - z^{n+1}}{1 - z} \right| \leq \frac{2}{|1 - z|}$$

if $|z| = 1$, $z \neq \underline{1}$.

Absolute Convergence:

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k|$

and we are done by Cauchy.

Theorem: $A = \sum a_n$, $B = \sum b_n$,

then $\sum a_n + b_n = A + B$ and $\sum c a_n = c \sum a_n$

Convolution: (product)

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Life is not simple: ☺

The convolution of two convergent series may diverge!

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$$a_n = \frac{(-1)^n}{\sqrt{n+1}} \qquad \sum a_n \text{ converges}$$

$$c_n = \sum_{k=0}^{n} a_k a_{n-k} = \sum_{k=0}^{n} \frac{(-1)^k \cdot (-1)^{n-k}}{\sqrt{k+1}\,\sqrt{n-k+1}} =$$

$$(-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\,\sqrt{n-k+1}}$$

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \le$$

$$\left(\frac{n}{2}+1\right)^2, \text{ so}$$

$$|c_n| \ge \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2} \not\to 0 \text{ as } n \to \infty$$

How was one supposed to know that $(n-k+1)(k+1) \le \left(\frac{n}{2}+1\right)^2$ via the arithmetic above?

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$$(n-k+1)+(k+1)=n+2$$

Let's rephrase: α, β positive real numbers, $\alpha+\beta=\gamma$. What is the max of $\alpha \cdot \beta$?

$$\alpha \cdot \beta = \alpha \cdot (\gamma - \alpha) = \gamma\alpha - \alpha^2$$

$$-(\alpha^2 - \alpha\gamma) = -\left(\left(\alpha - \frac{\gamma}{2}\right)^2 - \frac{\gamma^2}{4}\right) =$$

$$\frac{\gamma^2}{4} - \left(\alpha - \frac{\gamma}{2}\right)^2 \leq \frac{\gamma^2}{4}$$ and we are led straight to the arithmetic above.

Theorem (Life is OK! ☺)

a) $\sum a_n$ converges absolutely

b) $\sum_{n=0}^{\infty} a_n = A$ c) $\sum_{n=0}^{\infty} b_n = B$

d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $n = 0, 1, 2, \ldots$

Then $\sum_{n=0}^{\infty} c_n = A \cdot B$

⑧

Proof: $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$,

$$\beta_n = B_n - B$$

$$C_n = a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)$$

$$= a_0B_n + a_1B_{n-1} + \cdots a_nB_0$$

$$= a_0(B+\beta_n) + a_1(B+\beta_{n-1}) + \cdots + a_n(B+\beta_0)$$

$$= A_nB + \underbrace{a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0}_{\substack{\parallel \\ \gamma_n}}$$

It is enough to show that $\lim_{n\to\infty} \gamma_n = 0$ (why?)

Let $\alpha = \sum_{n=0}^{\infty} |a_n|$.

Let $\epsilon > 0$ be given. Choose $N \ni |\beta_n| \le \epsilon$ for $n \ge N$,

so $|\gamma_n| \le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N+1} + \cdots + \beta_n a_0|$

$$\le |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \epsilon\alpha$$

Keep N fixed and let $n \to \infty$, we see that

$$\overline{\lim_{n\to\infty}} |\gamma_n| \leq \epsilon\alpha \Rightarrow \lim_{n\to\infty} \gamma_n = 0.$$

<u>please study the rearrangement section on your own</u>

We start on continuity on Wednesday!