Math 265 H, Fall 2022, Oct 24

Theorem: $\sum a_n$ converges iff for every $\epsilon > 0$ $\exists N \ni \left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$.

Proof: This is just the Cauchy Criterion.

Theorem: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Proof: Take $m=n$ in the previous theorem.

Theorem: A series of non-negative terms coverges iff partial sums $\{s_n\}$ are bounded.

Proof: Theorem 3.14 implies this instantly.

②

Theorem: a) If $|a_n| \leq c_n$ for $n \geq N_0$ fixed

and if $\sum c_n$ converges, then $\sum a_n$ converges.

Proof: By Cauchy, given $\epsilon > 0$ $\exists$ $N \geq N_0 \ni$ $m \geq n \geq N \Longleftrightarrow$

$$\sum_{k=n}^{m} c_k \leq \epsilon.$$

This implies that

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \epsilon$$

and we are done!

b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof: Follows directly from a).

③

Theorem: If $0 \leq x < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geq 1$, the series diverges.

Proof: $S_n = \sum_{k=0}^{n} x^k =$

$$1 + x + x^2 + \dots + x^n$$

$$xS_n = \quad x + x^2 + \dots + x^n + x^{n+1}$$

$$S_n(1-x) = 1 - x^{n+1} \Rightarrow S_n = \frac{1-x^{n+1}}{1-x}$$

$$S_n \to \frac{1}{1-x} \text{ if } |x| < 1$$

If $x = 1$, we get $S_n = n \Rightarrow$ diverges

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Theorem: $a_1 \geq a_2 \geq \dots \geq 0$

Then $\sum_{n=1}^{\infty} a_n$ converges

iff $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges

<u>Proof:</u> $s_n = a_1 + a_2 + \dots + a_n$

$t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$

If $n < 2^k$,

$s_n \leq a_1 + (a_2 + a_3) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1})$

$\leq a_1 + 2a_2 + \dots + 2^k a_{2^k} = t_k$, so

$s_n \leq t_k$ ✓

If $n > 2^k$,

$s_n \geq a_1 + a_2 + (a_3 + a_4) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k})$

$\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \dots + 2^{k-1} a_{2^k}$

$= \frac{1}{2} t_k$, so $2s_n \geq t_k$

$\Updownarrow$

$s_n \geq \frac{t_k}{2}$

It follows that $s_n$ & $t_k$ are either both bounded or both unbounded, and we are done.

__Examples:__ $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$

e makes an appearance!

Definition: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Since $2^n \leq n!$, $n \geq 2$, the series converges.

Theorem: $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e$

__Proof:__ Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, $t_n = \left(1+\frac{1}{n}\right)^n$

$$t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+ \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

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Indeed,

$$\left(1+\frac{1}{n}\right)^n = 1^n \cdot \left(\frac{1}{n}\right)^0 + n \cdot 1^{n-1} \left(\frac{1}{n}\right) +$$

$$\binom{n}{2} 1^{n-2} \left(\frac{1}{n^2}\right) + \binom{n}{3} 1^{n-3} \frac{1}{n^3} +$$

$$\ldots$$

$$= 1 + 1 + \frac{n(n-1)}{n^2} \frac{1}{2!} +$$

$$\frac{n(n-1)(n-2)}{n^3} \frac{1}{3!} + \ldots$$

$$= 1 + 1 + \left(1-\frac{1}{n}\right)\frac{1}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!}$$

$$+ \ldots$$

It follows that

$t_n \le s_n$, so $\overline{\lim}\, t_n \le e$ (Theorem 3.19)

If $n \ge m$,

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$$

Let $n \to \infty$ (fixed $m$)

$$\Downarrow$$

$$\underline{\lim}\, t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$$

so $s_m \le \underline{\lim}\, t_n$, hence

$\lim s_m \le \underline{\lim}\, t_n$, so

$e \le \underline{\lim}\, t_n$ & we are done!