(1)

Math 265H, Fall 2022, October 17

Theorem: a) In any X metric space every convergent sequence is Cauchy.

b) If X is a compact metric space and if $\{p_n\}$ is Cauchy in X, then $\{p_n\}$ converges to some point of X.

c) In $\mathbb{R}^k$, every sequence converges iff it is Cauchy.

Proof: a) Since $p_n \to p$, given $\epsilon > 0$ $\exists N \ni$

$n \geq N$, $m \geq N$ $\Rightarrow |p_n - p_m| \leq |p_n - p| + |p_m - p|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

b) $\{p_n\}$ Cauchy in X compact.

$E_N = \{p_{N+1}, p_{N+2}, \dots, \}$ Then

$\lim \operatorname{diam}(\overline{E_N}) = 0$ by the previous result.

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Since $\overline{E_N}$ is closed, $\overline{E_N}$ is compact. Moreover,

$E_N \supset E_{N+1} \Rightarrow \overline{E_N} \supset \overline{E_{N+1}}$

The previous theorem, once again, shows that

$\exists!\, p$ that lies in every $\overline{E_N}$.

Let $\epsilon > 0$ be given. Then for N large enough,

$\operatorname{diam}(\overline{E_N}) < \epsilon$ if $N \geq N_0$, say.

Since $p \in \overline{E_N}$, $d(p,q) < \epsilon$ $\forall q \in \overline{E_N}$,

hence for every $q \in E_N$. It follows that

$d(p, p_n) < \epsilon$ for every $p_n \ni n \geq N_0$

$\Rightarrow p_n \xrightarrow[n \to \infty]{} p$.

c) $\{x_n\}$ Cauchy in $\mathbb{R}^k$. Define E_N as

above, w/ x_i in place of p_i. If N is

large enough, $\operatorname{diam}(E_N) < 1$. It follows

that $\{x_n\}$ is bounded (why?) Since every

(3)

Bounded set has a compact closure (in $\mathbb{R}^k$),

c) follows from b).

Definition: A metric space where every sequence converges is said to be complete.