(1)

Math 265H, October 12, 2022

Subsequences:

Consider $\{p_n\}$ sequence and consider

$n_1 < n_2 < \ldots < n_k < \ldots$

Then $\{p_{n_i}\}_i$ is called a subsequence of $\{p_n\}$

If $\{p_{n_i}\}$ converges, the limit is called a subsequential limit of $\{p_n\}$.

Example: $p_n = (-1)^n$ does not converge

But, $\{p_{2k}\}_{k=1}^{\infty}$ converges just fine!

Theorem:

a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence converges to a point in X.

(2)

b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof: Let $E = \{p_n\}$. If E is finite, nothing to prove.

If E is infinite, E has a limit point $p \in X$ (Theorem 2.37)

Choose $n_i \ni d(p, p_{n_i}) < \frac{1}{i}$

$\hookrightarrow \{p_{n_i}\} \longrightarrow p$ converges

Part b) follows from a) since Theorem 2.41

$\hookrightarrow$ every bounded subset of $\mathbb{R}^k$ lies in a compact subset of $\mathbb{R}^k$. Why?

③

Theorem: The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.

Proof: $E^* =$ subsequential limits of $\{p_n\}$
$q =$ limit point of E^*

We must show that $q \in E^*$.

Choose $n_1 \ni p_{n_1} \neq q$ (what if it does not exist?)

Let $\delta = d(q, p_{n_1})$. Assume that $n_1, n_2, \ldots, n_{i-1}$ have been chosen.

Since q is a limit point point of E^*,

$\exists\, x \in E^* \ni d(x, q) < 2^{-i}\delta$.

Since $x \in E^*$, $\exists\, n_i > n_{i-1} \ni d(x, p_{n_i}) < 2^{-i}\delta$.

Thus $d(q, p_{n_i}) \leq 2 \cdot 2^{-i}\delta$

$\Longrightarrow p_{n_i} \xrightarrow{\text{converges}} q$ ✓

④

Cauchy sequences:

A sequences $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\epsilon > 0$ $\exists N \ni d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Diameter: $E \subseteq X$

$\neq \emptyset$ metric space

$$S = \{d(p,q) : p \in E, q \in E\}$$

$$\sup(S) \equiv \text{diam}(E)$$

Observation: $\{p_n\} \subset X$ sequence;

$$E_N = \{p_N, p_{N+1}, \ldots\}$$

Then $\{p_n\}$ is Cauchy iff $\lim\limits_{N \to \infty} \text{diam}\, E_n = 0$.

(5)

Theorem: a) $E \subset X$, metric space

Then $\text{diam}\,\bar{E} = \text{diam}(E)$

b) If $\{K_n\}$ sequence of compact sets $\ni$

$K_n \supset K_{n+1}$ and if

$\lim_{n\to\infty} \text{diam}\,K_n = 0$,

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Proof: Since $E \subset \bar{E}$, $\text{diam}(E) \le \text{diam}(\bar{E})$

Fix $\epsilon > 0$, and choose $p \in \bar{E}$, $q \in \bar{E}$. $\exists\, p', q' \in E$

$\ni d(p,p') < \epsilon$, $d(q,q') < \epsilon$, so

$d(p,q) \le d(p,p') + d(p',q') + d(q',q)$

$< 2\epsilon + d(p',q') \le 2\epsilon + \text{diam}(E)$ ✓

So, $\text{diam}(\bar{E}) \le 2\epsilon + \text{diam}(E)$,

and we are done.

⑥

Proof of b)

$K = \bigcap_{1}^{\infty} K_n$. By Theorem 2.36,

$K \neq \emptyset$. If K has more than one point,

$\operatorname{diam}(K) > 0$. But $K_n \supset K$, so

$\operatorname{dim}(K_n) \geq \operatorname{diam}(K)$

$\Longrightarrow$ contradiction since

$\operatorname{diam}(K_n) \to 0$.