Math 265H, Oct 5, 2022

Convergent sequences: $\{p_n\} \in X \sim$ metric space

We say that $p_n \xrightarrow[n\to\infty]{X} p$ if for every $\epsilon > 0$ $\exists$

$N \ni$ $n \geq N$ implies that $d(p_n, p) < \epsilon$.

If p_n does not converge, it is said to diverge.

Theorem: a) $\underbrace{\{p_n\}}_{\text{sequence in } X}$ converges to $p \in X$ iff

every nhood of p contains all but finitely many elements of $\{p_n\}$.

b) If $p \in X$, $p' \in X$, $p_n \xrightarrow[n\to\infty]{X} p \Longrightarrow p = p'$

c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

d) If $E \subset X$ & p is a limit point of E, then $\exists$ sequence $\{p_n\}$ in $E \ni p = \lim_{n\to\infty} p_n$.

Proof: For any $\epsilon > 0$, $\exists N \ni$ if $n \geq N$, $d(p, p_n) < \epsilon$. This implies that $\leq N$ points of $\{p_n\}$ are outside the nhood of radius ϵ of p.

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Conversely, suppose that every nhood of p contains all but finitely many p_n's. Let $\epsilon>0$ be given. Then $\exists$ finitely many p_n's, say $p_{n_1}, p_{n_2}, \ldots p_{n_k}$, $n_1 < n_2 < \ldots < n_k$ are outside the ϵ nhood of $\underline{p}$.

Let $N = p_{n_k}$. Then for $n \geq N$, $d(p, p_n) < \epsilon$, so $\lim_{n\to\infty} p_n = p$ ✓

Proof of b) Let $\epsilon>0$ be given. By definition, $\exists N, N' \ni$ $n \geq N \Rightarrow d(p_n, p) < \frac{\epsilon}{2}$

$n \geq N' \Rightarrow d(p_n, p') < \frac{\epsilon}{2}$

Therefore, $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$

Since ϵ is arbitrary, $p = p'$ ✓

Proof of c) Let $\epsilon = 1$. Then $\exists N \ni$ $n \geq N \Rightarrow d(p, p_n) < 1$.

It follows that $d(p, p_n) \leq \max\{1, d(p, p_1), \ldots, d(p, p_N)\}$

Proof of d) For each $n>0$ $\exists$ $p_n \in E \ni d(p_n, p) < \frac{1}{n}$.

Given $\epsilon>0$, choose $N \ni N\epsilon > 1$. If $n > N$, $d(p, p_n) < \epsilon$, so $p_n \to p$.

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Theorem: Suppose $\{s_n\}, \{t_n\} \in \mathbb{C}$
$\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then

a) $\lim_{n\to\infty} (s_n + t_n) = s+t$

b) $\lim_{n\to\infty} c s_n = cs$, $\lim_{n\to\infty} c + s_n = c+s$ ($\tilde{c}$ any constant)

c) $\lim_{n\to\infty} s_n t_n = st$

d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ and $s \neq 0$

proof:

a) Given $\epsilon > 0$ $\exists$ N_1, N_2 $\ni$ $n \geq N_1 \Rightarrow |s_n - s| < \frac{\epsilon}{2}$,
$n \geq N_2 \Rightarrow |t_n - t| < \frac{\epsilon}{2}$

It follows that if $n \geq \max\{N_1, N_2\}$
$\Rightarrow |(t_n + s_n) - (t+s)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

b) Trivial

c) Observe that $s_n t_n - st =$

$$(s_n - s)(t_n - t) + s t_n + s_n t - 2st$$

$$= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$$

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Let $\epsilon > 0$ be given. Then $\exists N_1, N_2 \ni$

$n \geq N_1 \Rightarrow |s_n - s| < \sqrt{\epsilon}$

$n \geq N_2 \Rightarrow |t_n - t| < \sqrt{\epsilon}$

$\Rightarrow n \geq \max\{N_1, N_2\} \Rightarrow |(s_n - s)(t_n - t)| < \epsilon$,

so $\lim_{n \to \infty} ((s_n - s)(t_n - t)) = 0$

Combined w/ a) and b) we conclude that

$\lim_{n \to \infty} s_n t_n - st = 0$ and we are done.

Proof of d) Choose $m \ni |s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, and conclude that $|s_n| > \frac{1}{2}|s|$ for $n \geq m$.

Let $\epsilon > 0$ & find $N > m \ni$

$n \geq N \Rightarrow |s_n - s| < \frac{1}{2}|s|^2 \epsilon$

It follows that $n \geq N \Rightarrow$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{1}{\cancel{2}}|s|^2 \epsilon \cdot \frac{\cancel{2}}{|s|^2}$$

$= \epsilon$.

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Theorem:

a) $x_n \in \mathbb{R}^k$ $\quad x_n = (\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{k,n})$
Then x_n converges to $x = (\alpha_1, \dots, \alpha_k)$
iff $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$, $\quad 1 \le j \le k$

b) $x_n, y_n \in \mathbb{R}^k$, β_n real sequence, and
$x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then
$\lim_{n\to\infty} (x_n + y_n) = x + y$, $\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$,

$$\lim_{n\to\infty} \beta_n x_n = \beta x.$$

proof of a): Since $|\alpha_{j,n} - \alpha_j| \le |x_n - x|$,
one direction follows.

Conversely, let $\epsilon > 0$ be given. Then $\exists N \ni$
$$n \ge N \Longrightarrow |\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}, \quad 1 \le j \le k$$
Hence $n \ge N \Longleftrightarrow |x_n - x|$
$$= \left(\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2 \right)^{\frac{1}{2}} < \epsilon \checkmark$$

Part b) follows from a) and the previous result.