Monday, October 3 Math 265H

Theorem: If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof: Suppose not. Then each $q \in K$ has an nhood V_q containing no more than one point of E (i.e. if $q \in E$). No finite subcollection of $\{V_q\}$ can cover E, and the same is true for K since $E \subseteq K$. This contradicts compactness of K.

this is because E is infinite.

Theorem: If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ $\ni$ $I_n \supset I_{n+1}$, then $\bigcap_1^\infty I_n$ is not empty.

Proof: Write $I_n = [a_n, b_n]$ E = set of a_n's.
E is non-empty and bounded above by b_1.
Let $x = \sup E$. We have

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m, \text{ so}$$

$$x \leq b_m \text{ for each } m.$$

Clearly, $a_m \leq x$, so $x \in I_m$ for $m = 1, 2, 3, \ldots$

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<u>Corollary</u>: k positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $\forall n$, then $\bigcap_1^\infty I_n$ is not empty.

<u>Theorem</u>: Every k-cell is compact.

Proof: $I = k$-cell

$$= \{x \in \mathbb{R}^k : a_j \le x_j \le b_j,\ 1 \le j \le k\}$$

Let $\delta = \left(\sum_1^k (b_j - a_j)^2\right)^{\frac{1}{2}}$, ie $|x-y| \le \delta$ if $x, y \in I$

Suppose that $\exists$ cover $\{G_\alpha\}$ w/ no finite subcover. Let $c_j = \frac{a_j + b_j}{2}$ (midpoint)

The intervals $[a_j, c_j]$, $[c_j, b_j]$ determine 2^k k-cells Q_i whose union is I. At least one of them, say I_1, cannot be covered by a finite subcollection of $\{G_\alpha\}$.

Subdivide I_1 and run the same process.

We end up w/ $\{I_n\} \ni$

a) $I \supset I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$

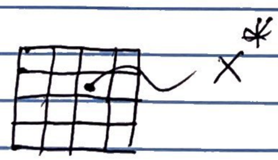
b) I_n is not covered by a finite subcollection of $\{G_\alpha\}$

c) If $x \in I_n$, $y \in I_n$, $|x-y| \le 2^{-n}\delta$

a) + theorem above $\Longrightarrow \exists\, x^* \in \bigcap_{1}^{\infty} I_n$.

Note that for some α, $x^* \in G_\alpha$. Since G_α is open, $\{y : |y-x^*| < r\} \subset G_\alpha$ for some $r > 0$. Choose $n \ni 2^{-n}\delta < r$ and note that

$I_n \subset G_\alpha$ by c) above, which contradicts b).



This result generalizes almost instantly to Heine-Borel theorem.

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Heine-Borel: If $E \subseteq \mathbb{R}^k$ has one of the following properties, then it has the other two.

i) E is closed and bounded

ii) E is compact

iii) Every infinite subset of E has a limit point in E.

Proof: If i) holds, put $E \subset \underset{k\text{-cell}}{\underline{T}}$

and ii) follows since closed subsets of compact sets are compact.

ii) $\Rightarrow$ iii) by the first result today.

It remains to show that iii) $\Rightarrow$ i)

If E is not bounded, $\exists$ sequence $\{x_n\}$ w/ $|x_n| > n \Rightarrow$ no limit point $\Rightarrow$ contradiction!

If E is not closed, $\exists$ $x_0 \in \mathbb{R}^k$ which is a limit point of E, but not a point in E. Construct $\{x_n\} \in E$ $\ni |x_0 - x_n| < \frac{1}{n}$; $S = \{x_n\}_{n=1}^{\infty}$

Then S is infinite, has x_0 as a limit point, and no other limit points, as we shall see in a moment.

It follows that if $y \in \mathbb{R}^k$, $y \neq x_0$, then

$$|x_n - y| \geq |x_0 - y| - |x_n - x_0|$$

why?

$$\geq |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|$$

for all but finitely many n.

This implies that y is not a limit point. Therefore, S has no limit points, so E must be closed if iii) holds.

Theorem (Weirstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

proof: Since E is bounded, $E \subset \underbrace{I}_{\text{some } k\text{-cell in } \mathbb{R}^k}$

Since we have shown that I is compact, E has a limit point by the first result we proved today.

We now turn to an alternate proof of uncountability of reals.

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Theorem: Let P be a non-empty perfect set in $\mathbb{R}^k$.
Then P is uncountable.

Proof: Since P has limit points, P is infinite.
Suppose that P is countable: $x_1, x_2, \ldots, x_n, \ldots$
Let $V_1 = \{y \in \mathbb{R}^k : |x_1 - y| < r\}$,

$\overline{V_1} = \{y \in \mathbb{R}^k : |x_1 - y| \leq r\}$

closure of V_1.

Suppose that V_n has been constructed so that $V_n \cap P \neq \emptyset$. Since every point of P is a limit point of P, $\exists$ V_{n+1} nhood $\ni$

i) $\overline{V_{n+1}} \subset V_n$

ii) $x_n \notin \overline{V_{n+1}}$

iii) $V_{n+1} \cap P \neq \emptyset$

inductive hypothesis satisfied

Let $K_n = \overline{V_n} \cap P$

$\underbrace{}_{\text{closed \& bounded, hence compact}}$

Since $x_n \notin K_{n+1}$, no point of P lies in $\bigcap_1^\infty K_n$.

Since $K_n \subset P$, $\bigcap_1^\infty K_n = \emptyset$. But each K_n is non-empty, and $K_n \supset K_{n+1}$, so we have a contradiction!

<u>Corollary:</u> Every interval $[a, b]$, $a < b$, is uncountable.

<u>Connected sets:</u>

<u>Definition:</u> $A, B \subseteq X$ (metric space) are separated if

$$A \cap \overline{B} = \overline{A} \cap B = \emptyset$$

$E \subset X$ is connected if $E \neq$ union of (non-empty) separated sets.

Example: $[0,1]$ & $(1,2)$ <u>are not</u> separated, but $(0,1)$ & $(1,2)$ <u>are</u> separated.

<u>Theorem</u>: A subset E of the real line $\mathbb{R}$ is connected iff it has the following property:

If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

proof: Suppose that $\exists\, x, y \in E$, $z \in (x, y) \ni z \notin E$.

Then $E = A_z \cup B_z$,

$$A_z = E \cap (-\infty, z), \quad B_z = E \cap (z, \infty)$$

$$x \in \qquad\qquad y \in$$

so A_z, B_z non-empty

Since $A_z \subset (-\infty, z)$, $B_z \subset (z, \infty)$,

A_z, B_z are separated. Hence E is not connected.

Conversely, suppose that E is not connected. Then

$$\exists\, A, B \ni \underbrace{A \cup B = E}_{\text{separated}}$$

Choose $x \in A$, $y \in B$, $x < y$ WLOG.

Let $z = \sup(A \cap [x, y])$, then $z \in \bar{A}$

$\downarrow$

$z \notin B$, so $x \leq z < y$.

If $z \notin A$, $x < z < y$ and $z \notin E$ ✓

If $z \in A$, then $z \notin \bar{B}$, so $\exists\, z_1$ w/ $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$ ✓