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September 19, 2022 Math 265H

Definition: A set X is called a metric space if $\exists\ d: X \times X \longrightarrow \mathbb{R}\ \ni$

a) $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$

b) $d(p,q) = d(q,p)$

c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

Example: $X = \mathbb{R}^k$, $d(x,y) = |x-y|$

Example: X = any non-empty set

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}.$$

check that the metric space axioms are satisfied.

$(a, b) = \{ x \in \mathbb{R} : a < x < b \}$

segment

$[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$

$x = (x_1, x_2, \dots, x_K) \ni \quad a_i \leq x_i \leq b_i \quad 1 \leq i \leq K$

is called a k-cell.

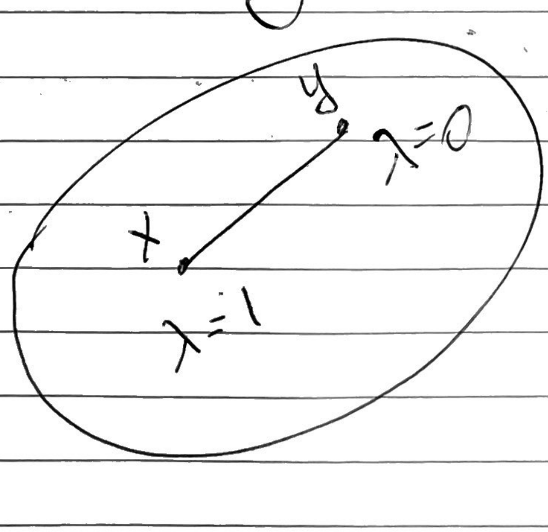
Open ball $B(x, r) = \{ y \in \mathbb{R}^K : |x - y| < r \}$

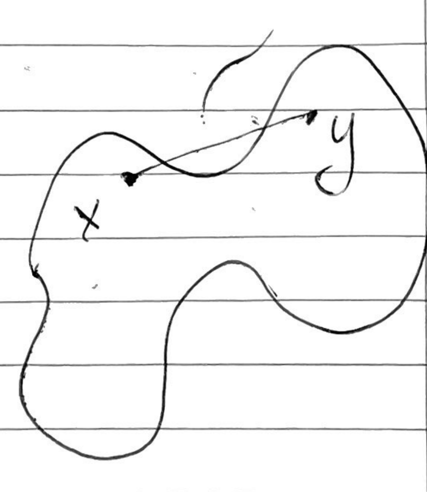
$E \subseteq \mathbb{R}^K$ is called convex if

$\lambda x + (1-\lambda) y \in E \quad 0 < \lambda < 1$

whenever $x, y \in E$

not convex





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Claim: Balls are convex

Proof: Let x = center of the ball, and let $y, z \in$ ball.

Then $|\lambda y + (1-\lambda)z - x| =$

$|\lambda(y-x) + (1-\lambda)(z-x)|$

$\leq \lambda|y-x| + (1-\lambda)|z-x|$

$< \lambda r + (1-\lambda)r = r$ ✓

Definition: X = metric space

a) Neighborhood of p: $N_r(p) = \{q \in X : d(p,q) < r\}$ for some $\underbrace{r}_{\text{radius}} > 0$.

b) p is a limit point of $E \subseteq X$ if every neighborhood of p contains a point $q \neq p \ni q \in E$.

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c) If $p \in E$ and p is not a limit point of E, then p is an isolated point of E.

d) E is closed if every limit point of E is a point of E.

e) A point p is an interior point of E if $\exists$ neighborhood of p contained in E.

f) E is open if every point of E is an interior point of E.

g) $E^c = \{x \in X : x \notin E\}$

complement of E

h) E is perfect if E is closed and if every point of E is a limit point of E.

i) E is bounded if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ $\forall p \in E$.

i) E is dense in X if every point of X is a limit point of E, or a point of E, or both.

run through examples of the definitions above

Theorem: Every neighborhood is a open set.

Let $E = N_r(p)$, $r>0$, and $q \in E$.

By definition, $\exists h>0 \ni d(p,q) = r-h$.

$\forall s \ni d(q,s) < h$,

$d(p,s) \leq d(p,q) + d(q,s) < r-h+h = r$

metric space triangle inequality

Therefore, $N_h(q) \subset N_r(p)$, so

q is an interior point.

<u>Theorem:</u> If p is a limit point of a set E, then every neighborhood of p contains <u>infinitely</u> many points of E.

<u>Proof:</u> We argue by contradiction. Suppose there is a neighborhood ~~of~~ N of p which contains only finitely many points of E.

Let $q_1, \dots, q_n \in N \cap E$

limit points $\quad q_i \neq p$

Let $r = \min\limits_{1 \leq m \leq n} d(p, q_m) > 0$

why?

By assumption, $N_r(p)$ contains no points of E – contradiction!

Corollary: A finite set has no limit points

⑦

Theorem: $\{E_\alpha\}$ finite or infinite collection of sets E_α.

Then $\underbrace{\left(\bigcup_\alpha E_\alpha\right)^c}_{A} = \underbrace{\bigcap_\alpha E_\alpha^c}_{B}$.

Proof: If $x \in A$, $x \notin \bigcup_\alpha E_\alpha$, which means that $x \in$ NONE of the set E_α, i.e. $x \in E_\alpha^c$ for every α, so

$x \in \bigcap_\alpha E_\alpha^c$. Thus $A \subset B$.

If $x \in B$, $x \in E_\alpha^c$ $\forall \alpha$, which means that $x \notin E_\alpha$ for any α, hence $x \notin \bigcup_\alpha E_\alpha$. We conclude that

$x \in \left(\bigcup_\alpha E_\alpha\right)^c$, so $B \subset A$.

It follows that $A = B$.

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<u>Theorem:</u> E is open iff its complement is closed.

<u>Proof:</u> Suppose that E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c, because E^c contains its limit points. Hence $\exists$ nhood N of $x \ni E^c \cap N$ is empty, i.e $N \subseteq E$. This implies that E is open.

<u>Conversely</u>, suppose that E is open. Let x be a limit point of E^c. Then every neighborhood of x contains a point of E^c, so that x is not an interior point of E. Since E is open, it follows that $x \in E^c$, which implies that E^c is closed.