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Math 265H, September 14, 2022

Let $f^{-1}(E) = \{x \in A : f(x) \in E\}$ $\quad E \subseteq B$

inverse image of E

If $y \in B$, $f^{-1}(y) = \{x \in A : f(x) = y\}$

If $f^{-1}(y)$ consists of at most one element of A.

Definition: If $\exists$ 1-1 of A onto B ($f(A) = B$), we say that $A \sim B$, or that A & B have the same cardinality.

We have i) $A \sim A$ ii) If $A \sim B$, then $B \sim A$
iii) If $A \sim B$ & $B \sim C$, then $A \sim C$.

i.e $\sim$ is an equivalence relation

Definition: $J_n = \{1, 2, \dots, n\}$
J = all integers (positive)

a) A is finite if $A \sim J_n$ for some n

b) A is infinite if A is not finite.

c) A is countable if $A \sim J$.

d) A is uncountable if A is neither finite nor countable.

e) A is at most countable if A is finite or countable

Example: $A = \{0, 1, 2, \dots \}$

Then $A \sim \mathbb{Z}$. Just map every number of the form $2k-1$, $k = \cancel{0}, 1, 2, \dots$ to k, and map every number of the form $2k$ to $-k$; 0 gets mapped to $\underline{\underline{0}}$.

Definition: A sequence is a function $f(n) = x_n$, $n \in J$

Theorem: Every infinite subset of a countable set A is countable.

Proof: Arrange A as a sequence

$$x_1, x_2, x_3, \dots, x_n, \dots$$

Arrange $E \subset A$ as a subsequence

$$x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots$$

$$n_1 < n_2 < \dots < n_k < \dots$$

and here is our bijection!

Theorem: Let $\{E_n\}$ be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof: Arrange E_n in a sequence $\{x_{nk}\}$, $k = 1, 2, \dots$

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \dots \\ x_{21} & x_{22} & x_{23} & x_{24} & \\ x_{31} & x_{32} & x_{33} & x_{34} & \end{array}$$

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Hence $S \subseteq J \times J$ & S is infinite.

Also, $J \times J \sim J$ by the scheme above.

It follows that S is countable.

Corollary: Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable.

Let $T = \bigcup_{\alpha \in A} B_\alpha$. Then T is at most countable.

Theorem: Let A = countable set, B_n = n-tuples $(a_1, a_2, \dots, a_n)$, $a_k \in A$. Then B_n is countable.

Proof: B_1 is countable. Suppose that B_{n-1} is countable.

$B_n = \{(b, a) : b \in B_{n-1}, a \in A\}$

For every fixed b, the set of pairs is countable, so B_n is a union of countable sets, hence countable. Proof complete by induction.

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Corollary: The set of all rational numbers is countable.

Proof: Apply the previous result w/ $n=2$ since rationals are of the form $\frac{b}{a}$, b, a integers. The set of pairs (a, b) and therefore the set of fractions $\frac{b}{a}$ is countable.

Is everything countable? No!

Theorem: The set of sequences of 1's & 0's is not countable.

Proof: Suppose otherwise. Then there is a bijection

$\sim$ 1's & 0's

$1: \quad a_{11} \; a_{12} \; a_{13} \; \ldots \; a_{1n} \; \ldots$

$2: \quad a_{21} \; a_{22} \; a_{23} \; \ldots \; a_{2n}$

$\vdots$

$K: \quad a_{K1} \; a_{K2} \; \ldots \; a_{Kn}$

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Consider a sequence of 1's & 0's

$b_1 \, b_2 \, b_3 \, \ldots \, b_n \ldots$ $\ni$

$\underline{\underline{b_K}}$ differs from $\underline{\underline{a_{KK}}}$

Then $b_1 \, b_2 \ldots b_n \ldots$ is not on the list above. Contradiction!