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Chapter 2, Math 174, Spring 2023

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diff at $a \in \mathbb{R}^n$ if

$\exists\ \underbrace{\lambda}_{\text{linear}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \ni$

norm

$$\lim_{\substack{h \to 0 \\ \in \\ \mathbb{R}^n}} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

Theorem: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diff at $a \in \mathbb{R}^n$,

$\exists!$ linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

unique

$$\ni \lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

Proof: Suppose $\underbrace{\mu}_{\text{linear}} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\ni \lim_{h \to 0} \left| \frac{f(a+h) - f(a) - \mu(h)}{|h|} \right| = 0.$$

(2)

If $d(h) = f(a+h) - f(a)$, then

$$\lim_{h \to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} =$$

$$\lim_{h \to 0} \frac{|\lambda(h) - d(h) + d(h) - \mu(h)|}{|h|}$$

$$\leq \lim_{h \to 0} \frac{|\lambda(h) - d(h)|}{|h|} + \lim_{h \to 0} \frac{|d(h) - \mu(h)|}{|h|}$$

$$= 0$$

Let $h = tx$, $x \neq 0$

Then $0 = \lim_{t \to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} =$

$$\frac{|\lambda(x) - \mu(x)|}{|x|} \longrightarrow \lambda = \mu$$

(3)

To get a feel for this, suppose that

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f(x) = Ax$, $A =$ $m \times n$ matrix

Then f is diff as we can take $\lambda = \underline{\underline{A}}$.

Now consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x, y) = xy$$

To see that f is diff at (a, b), consider

$$\lim_{(h,k) \to (0,0)} \frac{|(a+h)(b+k) - ab - \lambda(h,k)|}{|(h,k)|}$$

$$= \lim_{(h,k) \to (0,0)} \frac{|ak + bh - \lambda(h,k)|}{|(h,k)|}$$

Evidently, taking $\lambda(h,k) = ak + bh$ does the job.

In the definition of differentiability, let $Df(a) = \lambda$.
The underlying matrix is called the **Jacobian matrix**.

Chain rule: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diff at a,
$g: \mathbb{R}^m \longrightarrow \mathbb{R}^p$ diff at $f(a)$, then
$g \circ f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ diff at a, and

$$D(g \circ f)(a) = \underbrace{Dg(f(a)) \circ Df(a)}_{\text{composition of linear transformations}}$$

Proof: $b = f(a)$, $\lambda = Df(a)$
$\mu = Dg(f(a))$

Define
i) $\varphi(x) = f(x) - f(a) - \lambda(x-a)$
ii) $\psi(y) = g(y) - g(b) - \mu(y-b)$
iii) $\rho(x) = g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x-a)$

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Theorem: $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ diff at a,
$g : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ diff at $f(a)$, then
$g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ diff at a, and

$$D(g\circ f)(a) = Dg(f(a)) \circ Dg(a)$$

Proof: $b = f(a)$, $\lambda = Df(a)$

$\mu = Dg(f(a))$

$$\varphi(x) = f(x) - f(a) - \lambda(x-a)$$

$$\psi(y) = g(y) - g(b) - \mu(y-b)$$

$$\rho(x) = g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a)$$

Then $\displaystyle\lim_{x\to a} \frac{|\varphi(x)|}{|x-a|} = 0$, $(*)$

$$\lim_{y\to b} \frac{|\psi(y)|}{|y-b|} = 0 \quad (**)$$

We want: $\displaystyle\lim_{x\to a} \frac{|\rho(x)|}{|x-a|} = 0.$

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$$\rho(x) = g(f(x)) - g(b) - \mu(\lambda(x-a))$$
$$= g(f(x)) - g(b) - \mu(f(x) - f(a) - \varphi(x))$$
$$= [g(f(x)) - g(b) - \mu(f(x) - f(a))] + \mu(\varphi(x))$$
$$= \psi(f(x)) + \mu(\varphi(x))$$

We must show that (***)

$$\lim_{x \to a} \frac{|\psi(f(x))|}{|x-a|} = 0, \text{ and}$$

$$\lim_{x \to a} \frac{|\mu(\varphi(x))|}{|x-a|} = 0$$

follows from (*) and exercise 1.10

Now, (**) $\Rightarrow$ if $\epsilon > 0$,

$|\psi(f(x))| < \epsilon |f(x) - b|$ if $|f(x) - b| < \delta$,
which, in turn, holds if $|x-a| < \delta_1$ for some δ_1.

Then $|\psi(f(x))| < \epsilon |f(x) - b|$

$= \epsilon |\varphi(x) + \lambda(x-a)|$

$\leq \epsilon |\varphi(x)| + \epsilon M |x-a|$

problem 1.10

(~~***~~) follows instantly.

Theorem:

i) If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a constant function, then $Df(a) = 0$.

ii) If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, then $Df(a) = f$.

iii) If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then f is diff at $a \in \mathbb{R}^n$ iff each f^i is, and $Df(a) = (Df^1(a), \ldots, Df^m(a))$

Thus $f'(a)$ is the $m \times n$ whose i'th row is $(f^i)'(a)$.

⑧

iv) If $s : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by

$$s(x,y) = x+y, \text{ then}$$
$$Ds(a,b) = s.$$

v) If $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$; $p(x,y) = xy$,

then $Dp(a,b)(x,y) = bx + ay$

Thus $p'(a,b) = (b,a)$.

Proof: i) is immediate

ii) $\displaystyle\lim_{h \to 0} \frac{|f(a+h) - f(a) - f(h)|}{|h|}$

$$= \lim_{h \to 0} \frac{|f(a)+f(h) - f(a) - f(h)|}{|h|} = 0$$

iii) If each f^i is diff at a and

$\lambda = (Df^1(a), \dots, Df^m(a))$, then

$$f(a+h) - f(a) - \lambda(h) =$$
$$(f^1(a+h) - f^1(a) - Df^1(a)(h), \dots,$$
$$f^m(a+h) - f^m(a) - Df^m(a)(h))$$

The conclusion follows by the triangle inequality.

If f is diff at a, then $f^i = \pi^i \circ f$, so

projection

each f^i is diff by above and the chain rule.

iv) follows from ii)

v) Let $\lambda(x,y) = bx + ay$. Then

$$\lim_{(h,k)\to(0,0)} \frac{|p(a+h, b+k) - p(a,b) - \lambda(h,k)|}{|(h,k)|} = \lim_{(h,k)\to 0} \frac{|hk|}{|(h,k)|}$$

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We have

$$|hk| \leq \begin{cases} |h|^2, & \text{if } |k| \leq |h| \\ |k|^2, & \text{if } |k| \geq |h| \end{cases}$$

It follows that $|hk| \leq |h|^2 + |k|^2$.

Therefore, $\dfrac{|hk|}{\|(h,k)\|} \leq \dfrac{h^2+k^2}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2} \xrightarrow[(h,k)\to 0]{} 0$

<u>Corollary</u>: If $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ diff at a, then

$$D(f+g)(a) = Df(a) + Dg(a)$$

$$D(fg)(a) = g(a)\, Df(a) + f(a)\, Dg(a)$$

If $g(a) \neq 0$, then

$$D(f/g)(a) = \frac{g(a)\, Df(a) - f(a)\, Dg(a)}{|g(a)|^2}$$

Partial derivatives

It is time to actually start computing derivatives.

If $f : \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$,

$$\lim_{h \to 0} \frac{f(a^1, \dots, a^i + h, \dots, a^n) - f(a^1, \dots, a^n)}{h}$$

is called the i'th partial derivative at a.

$$\frac{\partial f}{\partial x^i}(a) \quad \text{or} \quad D_i f(a)$$

problem 3.28

Define $D_{i,j} f(x) = D_j (D_i f)(x)$

Theorem: If $D_{i,j} f$ and $D_{j,i} f$ are continuous in an open set containing a, then

$$D_{i,j} f(a) = D_{j,i} f(a).$$

Theorem: Let $A \subseteq \mathbb{R}^n$. If the max or min of $f: A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then

$$D_i f(a) = 0.$$

Proof: Follows from the one-variable case.

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}^m$ diff at a, then $D_j f^i(a)$ exists for $1 \leq i \leq m$, $1 \leq j \leq n$, and $f'(a)$ is the $m \times n$ matrix $\{D_j f^i(a)\}_{i,j}$

Proof: Suppose that $m = 1$, so $f: \mathbb{R}^n \to \mathbb{R}$

Define $h: \mathbb{R} \to \mathbb{R}^n$ by

$$h(x) = (a^1, \ldots, x, \ldots, a^n),$$ with x in the jth place. Then

$$D_j f(a) = (f \circ h)'(a^j).$$ Hence, by chain rule,

$$(f \circ h)'(a^j) = f'(a) \cdot h'(a^j)$$

(13)

$$= f'(a) \cdot \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ j'th place}$$

Since $(f \circ h^j)'(a^j)$ has the single entry $D_j f(a)$, $D_j f(a)$ exists and is the j'th entry of the $1 \times n$ matrix $f'(a)$. The result follows by Theorem 2.3.

<u>Theorem</u>: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $Df(a)$ exists if all $D_j f^i(x)$ exist in an open set containing a and if each function $D_j f^i$ is continuous at a.

Proof: It suffices to consider the case $m=1$, so that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then

$$f(a+h) - f(a) = f(a^1+h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n)$$
$$+ f(a^1+h^1, a^2+h^2, a^3, \ldots, a^n) - f(a^1+h^1, a^2, \ldots, a^n)$$
$$+ \ldots$$

(14)

$+ \dots +$

$f(a^1+h^1, \dots, a^n+h^n) - f(a^1+h^1, \dots, a^{n-1}+h^{n-1}, a^n)$.

Observation: $D_1 f$ = derivative of

$$g(x) = f(x, a^2, \dots, a^n)$$

By MVT,

$$f(a^1+h^1, a^2, \dots, a^n) - f(a^1, \dots, a^n)$$

$$= h^1 \cdot D_1 f(b_1, a^2, \dots, a^n) \quad \text{for}$$

some $b^1 \in (a^1, a^1+h^1)$.

The ith term is equal to

$$h^i \cdot D_i f(a^1+h^1, \dots, a^{i-1}+h^{i-1}, b_i, \dots, a^n)$$

$$= h^i D_i f(c_i) \quad \text{for some } c_i.$$

Then

$$\lim_{h \to 0} \frac{\left| f(a+h) - f(a) - \sum_{i=1}^{n} D_i f(a) \cdot h^i \right|}{|h|} =$$

$$= \lim_{h \to 0} \frac{\left| \sum_{i=1}^{n} [D_i f(c_i) - D_i f(a)] \cdot h^i \right|}{|h|}$$

$$\leq \lim_{h \to 0} \sum_{i=1}^{n} |D_i f(c_i) - D_i f(a)| \frac{|h^i|}{|h|}$$

$$\leq \lim_{h \to 0} \sum_{i=1}^{n} |D_i f(c_i) - D_i f(a)| = 0$$

by continuity!

<u>Theorem:</u> Let $g_1, g_2, \ldots, g_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously diff at a, and let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be diff at $(g_1(a), \ldots, g_m(a))$. Define $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $F(x) = f(g_1(x), \ldots, g_m(x))$. Then

$$D_i F(a) = \sum_{j=1}^{m} D_j f(g_1(a), \ldots, g_m(a)) \cdot D_i g_j(a)$$

Example: $g_1(x,y) = x+y \qquad g_2(x,y) = xy$

$f(x,y) = x^2 + y^2$

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$$f(g_1(x,y),\ g_2(x,y)) = F(x,y)$$

$$= g_1^2(x,y) + g_2^2(x,y) =$$

$$(x+y)^2 + (xy)^2 = x^2 + 2xy + y^2 + x^2y^2$$

$$D_1F(a,b) = 2a + 2b + 2ab^2 \checkmark$$

$$D_1 f(x,y) = 2x \qquad D_2 f(x,y) = 2y$$

$$D_1 g_1(a,b) = 1 \qquad D_1 g_2(a,b) = b$$

So, $\displaystyle\sum_{j=1}^{2} D_j f(g_1(a,b), g_2(a,b)) \cdot D_1 g_j(a,b)$

$$= 2g_1(a,b)\cdot 1 + 2g_2(a,b)\cdot b$$

$$= 2(a+b) + 2(ab)\cdot b$$

$$= 2a + 2b + 2ab^2 \checkmark$$

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Let us now prove the theorem above.

$F = f \circ g$, w/ $g = (g_1, \ldots, g_m)$

By chain rule,

$F'(a) = f'(g(a)) \cdot g'(a) =$

$$(D_1 f(g(a)), \ldots, D_m f(g(a))) \cdot \begin{pmatrix} D_1 g_1(a) & \cdots & D_n g_1(a) \\ \vdots & & \\ D_1 g_m(a) & \cdots & D_n g_m(a) \end{pmatrix}$$

Observe that $D_i F(a)$ is the ith entry of the left-hand side of the equation, while

$$\sum_{j=1}^{m} D_j f(g_1(a), \ldots, g_m(a)) \cdot D_i g_j(a)$$

is the ith entry of the right-hand side.

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Inverse functions:

Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$ and $f'(a) \neq 0$

continuously diff near $\underline{\underline{a}}$

Then f is 1-1 in a small nhood, so f^{-1} exists. Moreover,

$f \circ f^{-1}(x) = x$, so

$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$, which implies that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In higher dimensions, this is much harder!

Lemma: $A \subseteq \mathbb{R}^n$ rectangle, and let $f: A \to \mathbb{R}^n$ continuously differentiable.
If there is $M \ni |D_j f^i(x)| \leq M \ \forall x$ in A^o

interior,

then $|f(x) - f(y)| \leq n^2 M |x-y|$

$\forall x, y \in A$

Lipschitz property

Proof: We have

$$f^i(y) - f^i(x) = \sum_{j=1}^{n} \Big[f^i(y^1, \dots, y^j, x^{j+1}, \dots, x^n) - f^i(y^1, \dots, y^{j-1}, x^j, \dots, x^n) \Big]$$

MVT

$$= \sum_{i=1}^{n} (y^j - x^j) \cdot D_j f^i(z_{ij})$$

for some z_{ij}.

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It follows that

$$|f^i(y)-f^i(x)| \le \sum_{i=1}^{n} |y^j-x^j|\cdot M \le$$

$$nM|y-x|,$$ since

$$|y^j-x^j| \le |y-x|.$$

It follows that

$$|f(y)-f(x)| \le \sum_{i=1}^{n} |f^i(y)-f^i(x)| \le$$

$$n^2 M\cdot|y-x|$$ ✓

Inverse Function Theorem:

Suppose $f:\mathbb{R}^n \to \mathbb{R}^n$ continuously diff in an open set containing a, and $\det f'(a)\ne 0$. Then there is an open set V containing a and an open set W containing $f(a)$ such that $f: V \to W$ has a continuous inverse

$f^{-1}: W \longrightarrow V$ differentiable, and $\forall y \in W$ satisfies

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}.$$

Proof: Let $\lambda = Df(a)$. Then λ is non-singular, since $\det f'(a) \neq 0$.

Now $D(\lambda^{-1} \circ f)(a) \underset{\text{chain rule}}{=} D(\lambda^{-1})(f(a)) \circ Df(a) =$

$= \underbrace{\lambda^{-1}}_{\text{diff of linear is linear!!!}} \circ Df(a) \quad = \underline{\underline{\text{identity}}}!$

If the result holds for $\lambda^{-1} \circ f$, it holds for f!
So we may assume that $\lambda =$ identity, i.e

$$\underline{\underline{\lambda(h) = h}}.$$

With this reduction in tow, let's see what would happen if $f(a+h) = f(a)$ for some $\underline{h}$.

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We have

$$\frac{\left| f(a+h) - f(a) - \lambda(h) \right|}{|h|} = \frac{|h|}{|h|} = 1$$

On the other hand,

$$\lim_{h \to 0} \frac{\left| f(a+h) - f(a) - \lambda(h) \right|}{|h|} = 0,$$

by assumption,

so $f(x) \neq f(a)$ for $x \neq a$, x arbitrarily close to $\underline{a}$.

Therefore, $\exists$ closed rectangle U containing a in its interior such that

1) $f(x) \neq f(a)$ if $x \in U$, $x \neq a$.

2) $\det f'(x) \neq 0$ for $x \in U$. this is just continuity

3) $|D_j f^i(x) - D_j f^i(a)| < \frac{1}{2n^2}$ $\forall i, j$, $x \in U$.

also continuity

We now apply 3 and Lemma 2-10 to $g(x) = f(x) - x$ to see that for $x_1, x_2 \in U$, then

$$|f(x_1) - x_1 - (f(x_2) - x_2)| \leq \underbrace{\tfrac{1}{2}}_{n=1 \text{ in this case,,,}} |x_1 - x_2|.$$

Now a small algebraic trick. We have

$$|x_1 - x_2| - |f(x_1) - f(x_2)| \leq$$

$$|f(x_1) - x_1 - (f(x_2) - x_2)| \leq \tfrac{1}{2} |x_1 - x_2|, \text{ so}$$

4) $|x_1 - x_2| \leq 2 |f(x_1) - f(x_2)| \quad \forall\, x_1, x_2 \in U.$

Let ∂U = boundary of U. It is closed and bounded, so it is compact. Therefore, $f(\partial U)$ is compact, and does not contain $f(a)$.

It follows that

$$|f(x) - f(a)| \geq d \quad \text{for } x \in \partial U.$$

Let $W = \{ y : |y - f(a)| < \frac{d}{2} \}$

If $y \in W$, $x \in \partial U$, then

5) $|y - f(a)| < |y - f(x)|.$

<u>Goal:</u> For any $y \in W$, $\exists!\ x \in U^{o}$ $\ni$ $f(x)=y$.

interior

To prove this, let $g : U \longrightarrow \mathbb{R}$:

$$\underbrace{g(x) = |y - f(x)|^2}_{\text{continuous, so it has a minimum on } U.} = \sum_{i=1}^{n} (y^i - f^i(x))^2$$

If $x \in \partial U$, then ⑤ tells us that $g(a) < g(x)$.

$\exists\, g$ minimum is not achieved on ∂U

By Theorem 2.6, $\exists\ x \in U^{o} \ni D_j g(x) = 0\ \forall j$,

so $$\sum_{i=1}^{n} 2(y^i - f^i(x)) \cdot D_j f^i(x) = 0 \quad \forall j$$

The matrix $\{D_j f^i\}$ has non-zero determinant, so, $y^i - f^i(x) = 0 \quad \forall i$, i.e $y^i = f^i(x)$,

so $y = f(x)$.

This proves the existence of x. The uniqueness follows from (4.)

What have we shown?

Let $V = U^0 \cap f^{-1}(W)$. Then

$f : V \to W$ has an inverse

$f^{-1} : W \to V$. It is useful to rewrite (4) as

$$(6) \quad |f^{-1}(y_1) - f^{-1}(y_2)| \leq 2|y_1 - y_2|, \quad y_1, y_2 \in W$$

$\Rightarrow f^{-1}$ is continuous!

We must prove that f^{-1} is differentiable. We already have a bag of tricks for this. Let $\mu = Df(x)$. We will show that f^{-1} is differentiable and the derivative is $\underline{\underline{\mu^{-1}}}$.

We write

$$f(x_1) = f(x) + \mu(x_1 - x) + \varphi(x_1 - x) \quad w/$$

$$\lim_{x_1 \to x} \frac{|\varphi(x_1 - x)|}{|x_1 - x|} = 0.$$

Therefore, $\mu^{-1}(f(x_1) - f(x)) = x_1 - x + \mu^{-1}(\varphi(x_1 - x))$

RECALL: Every $y_1 \in W = f(x_1)$ for some $x_1 \in V$

It follows that

$$f^{-1}(y_1) = f^{-1}(y) + \mu^{-1}(y_1 - y) - \mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y))),$$

so matters have been reduced to showing that

$$\lim_{y_1 \to y} \frac{|\mu^{-1}(\varphi(f^{-1}(y_1) - f^{-1}(y)))|}{|y_1 - y|} = 0.$$

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By the ubiquitous problem 1.10, it is enough to show that

$$\lim_{y_1 \to y} \frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y|} = 0.$$

We have

$$\frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|y_1 - y_2|} =$$

$$\frac{|\varphi(f^{-1}(y_1) - f^{-1}(y))|}{|f^{-1}(y_1) - f^{-1}(y)|} \quad \frac{|f^{-1}(y_1) - f^{-1}(y)|}{|y_1 - y_2|}$$

$\downarrow 0$ since $f^{-1}(y_1) - f^{-1}(y) \to 0$ as $y_1 \to y$

bounded by 2 by (6)

and we are done!

Implicit functions:

Consider $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$,
the unit circle.

Let's work near $(x_0, y_0) = (0,1)$.
Then $y = \sqrt{1-x^2}$ (positive square root)

Let $f(x) = \sqrt{1-x^2}$

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}$$ exists near $x=0$

But near $x=1$, we have a problem. (dividing by 0!)

Calc 1 perspective:

$x^2+y^2=1$ $\quad 2x + 2y \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{x}{y},$$ OK as long as $y \neq 0$

We shall develop methods for taking the equation $f(x,y)=0$ and expressing $y=g(x)$ when this is possible.

<u>Theorem:</u> $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ cont diff in an open set containing (a,b); w/ $f(a,b)=0$.

Let M be the $m \times m$ matrix
$$\{D_{n+j} f^i (a,b)\} \quad 1 \leq i,j \leq m$$

If $\det M \neq 0$, $\exists\, A \subseteq \mathbb{R}^n$ open, and $B \subseteq \mathbb{R}^m$ open containing b, w/ following property: for each $x \in A$ $\exists!\, g(x) \in B \ni f(x, g(x)) = 0$. The function g is differentiable.

<u>Example:</u> $n=m=1$ $\quad f(x,y) = x^2+y^2-1$

$M = 1\times 1$ matrix $D_2 f(a,b) = 2y$, so we want $y \neq 0$, as we discussed above!

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$n=2 \quad m=1$

$f(x,y) = f(x^1, x^2, y) =$

$(x^1)^2 + (x^2)^2 + y^2 - 1$

$M = |x1|$ again

$D_3 f(x,y) = 2y$ and we want

$y \neq 0$ again!

Proof: Define $F: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$F(x,y) = (x, f(x,y))$

Then $\det F'(a,b) = \det M \neq 0$

why? write out every step!

By Inverse Function Theorem, $\exists$ W open in $\mathbb{R}^n \times \mathbb{R}^m$ containing $F(a,b) = (a,0)$ and an open set $A \times B$ containing (a,b) $\ni$

$F: A \times B \longrightarrow W$ has a diff

inverse $h: W \longrightarrow A \times B$

Observe that $h(x,y) = (x, K(x,y))$

diff

since F is of that form.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$

$\pi(x,y) = y$; $\pi \circ F = f$.

It follows that

$$f(x, K(x,y)) = f \circ h(x,y) = (\pi \circ F) \circ h(x,y)$$
$$= \pi \circ (F \circ h)(x,y) = \pi(x,y) = y.$$

Thus $f(x, K(x,0)) = 0$, so we may take $g(x) = K(x,0)$.

We will need the following generalization.

same proof as above

Theorem: $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ cont diff in an open set containing a, $p \leq n$. If $f(a) = 0$ and $\{D_j f^i(a)\}$ has rank p, then there is an open set $A \subseteq \mathbb{R}^n$ containing a, and a diff function $h : A \longrightarrow \mathbb{R}^n$ w/ diff inverse $\ni$ $f \circ h(x^1, \ldots, x^n) = (x^{n-p+1}, \ldots, x^n)$